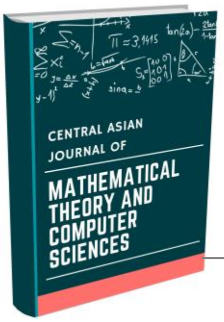

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Definition of Different Schemes for Calculation of General Solutions

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Abstract:
This article provides information on generalized solutions and how to calculate them. It is impossible to find a definite solution to some mathematical and physical problems. So, they would only resort to this as a last resort. In this paper, we will consider the problems of liquid and gas dynamics by solving them using an artificial viscosity scheme of a generalized solution and comparing it with a concrete solution.

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INTRODUCTION

In the construction of many differential schemes, it is assumed that there is a sufficiently smooth solution to the differential boundary value problem, and the products of the differential equation are replaced by approximately differential relations. However, differentiating functions is not enough to describe many important physical processes. For example, physical experiments have shown that when describing the distribution of pressure, density, and temperature in a non-viscous, supersonic gas stream, functions such as jump-shock waves appear. Jumping can occur over time, even when the initial conditions are smooth.

Differential boundary value problems corresponding to them do not have smooth solutions. In this sense, it is necessary to expand the concept of solution and introduce the concept of a generalized solution in some natural way, so that they have a break. There are two main ways to do this. The first method is to write the laws of physical conservation (mass, momentum, energy, etc.) in integral form, not in differential form. In this case, they also make sense for discontinuous functions, which cannot be differentiated, but can be integrated.

In the second method, terms are artificially introduced into the differential equation so that the equation has smooth solutions. These artificially introduced limits have a small viscosity value for hydrodynamic problems, smoothing out flow interruptions. Then the coefficients in front of the "viscous" limits tend to zero, and the limit to which the solution is sought is taken as the generalized solution of the initial problem.

MAIN PART

Let's look at the definition of a generalized solution and how to calculate it in the following Cauchy problem:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,0<t<T,-\infty<x<\infty\\&u(x,0)=\psi(x),-\infty<x<\infty\end{aligned}\right\} \quad (1)$$

This problem is a simple model of gas dynamics equations that can be solved continuously under smooth initial conditions.

Let's look at the mechanism of interruptions.

We initially assume that problem (1) has a smooth $u(x,t)$ solution. We introduce the lines

$$\frac{dx}{dt}=u(x,t), \quad (2)$$

$x=x(t)$ defined by this equation. These lines are called the characteristics of the equation $u_t+uu_x=0$. For each characteristic $x=x(t)$, the function $u(x,t)$ can be considered only as a function of the variable t :

$$u(x,t)=u[x(t),t]=u(t)$$

In this case, this equation

$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$ is valid.

Therefore, the solution does not change along the characteristic, that is, $u(x,t)=const$. However, according to Equation (2), the condition $u=const$ means that the characteristic $x=ut+x_0$ is a straight line. Here x_0 is the abscissa of the point from which the characteristic emerges $(x_0,0)$, and $u=\psi(x_0)$ is the angular coefficient of the slope of the abscissa to the Ot axis. By giving the initial function $u(x,0)=\psi(x)$, the image of the characteristic is visualized and the value of the solution in the half-plane $t>0$ is determined.

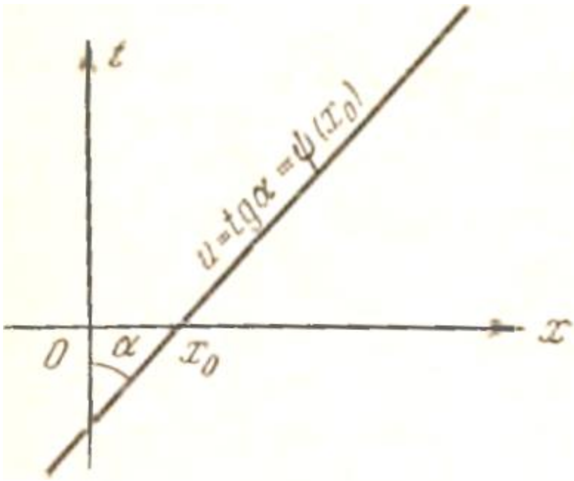


Figure 1

It should be noted that the assumption of the existence of a smooth solution $u(x,t)$ means that the characteristics do not intersect, because each characteristic carries the value of its solution to the point of intersection, and the solution is not a one-valued function. For a monotonous growing function $\psi(x)$, as the x_0 increases, the α angle increases and the characteristics do not intersect.

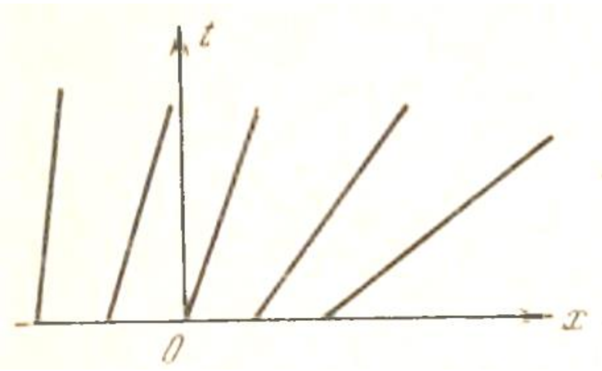


Figure 2

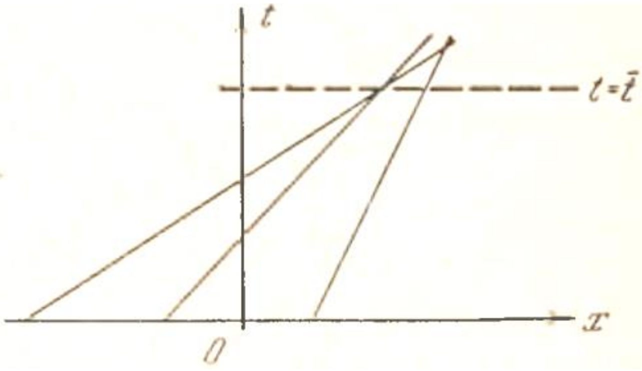


Figure 3

However, when the function $\psi(x)$ decreases, the characteristics converge and their intersection is inevitable, while $\psi(x)$ does not depend on the smoothness of the function. From the moment $t=\bar{t}$ when at least two characteristics intersect, there is no smooth solution to problem (1) (Figure 3).

The graphs of the function $u=u(x,t)$ at time $t=0,\frac{1}{2}\bar{t}$ and $\bar{t}$ are shown in Figure 4.

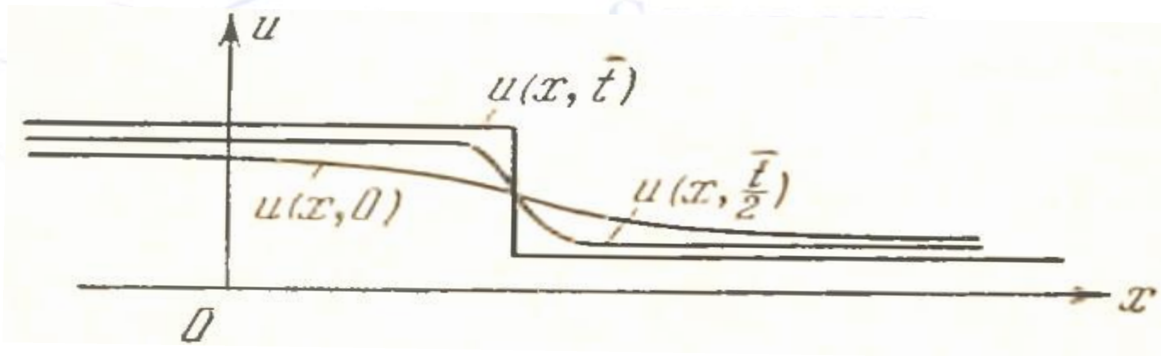


Figure 4

Here are the Green formulas used to determine the generalized solution of Problem (1). Let us consider the sphere D with boundary G in the plane Oxt, and let the functions $\Phi_1(x,t)$ and $\Phi_2(x,t)$ have continuous special products in the sphere D with its boundary G. In this case, the following Green formula is appropriate:

$$\iint_D \left(\frac{\partial \Phi_1}{\partial t}+\frac{\partial \Phi_2}{\partial x}\right)dxdt=\oint_G \Phi_1 dx-\Phi_2 dt, \qquad (3)$$

The expression $\frac{\partial \Phi_1}{\partial t}+\frac{\partial \Phi_2}{\partial x}$ is the divergence of the vector $\Phi=\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$.

(3) The Green formula means that the integral vector field Φ is equal to the integral of the flow vector Φ obtained along the G boundary.

We will now describe a generalized solution. We write the differential problem (1) in divergent form:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0 \qquad (4)$$

We integrate both sides of Equation (4) in any area D lying in the semi-plane $t\geq 0$ and form the following equation:

$$0=\iint_D\left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right]dxdt=\oint_G udx-\frac{u^2}{2}dt.$$

Thus, each differentiable solution of equation (4) satisfies this integral equation:

$$\oint_G udx-\frac{u^2}{2}dt=0 \qquad (5)$$

where G- $t>0$ is an arbitrary contour lying in the half-plane. Relation (5) represents any law of conservation. The current of the vector $\begin{pmatrix}u\\ u^2/2\end{pmatrix}$ through an arbitrary closed loop is zero. We show that the opposite is also true. If the smooth function satisfies the law of integral conservation (5) on an arbitrary contour G, then the equation (4) holds for each point $(x_0,t_0),t_0>0$.

Assume the inverse and let

$$\left.\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right|_{\substack{x=x_0\\ t=t_0}}>0$$

be at some point (x_0,t_0) for clarity. In this case, according to the continuity of the function, we find a small circle D bounded by the boundary G at the center (x_0,t_0), at all points of which there is $u_t+\left(\frac{u^2}{2}\right)_x>0$,

and the following relation is valid:

$$0=\oint_G udx-\frac{u^2}{2}dt=\iint_D\left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right]dxdt>0.$$

The resulting contradiction $0>0$ proves that for a smooth function $u(x,t)$ it follows from equation (5) (4), that is, equations (4) and (5) are equally strong. However, when the function $u(x,t)$ has a break, the differential equation (1) or (4) loses its meaning on the break line, while the integral condition (5) does not lose its meaning. Therefore, the generalized solution of equation (4) is an arbitrary fractional differentiable function that satisfies condition (5) when the contour G is arbitrarily chosen in the half-plane $t\geq 0$.

The break line of the solution and its condition. Let there be a line $x\equiv x(t)$ in the field in which the solution is sought, so that the generalized solution $u(x,t)$ has the first type of interruption. Approaching this line from left or right, we have the functions

$$u(x,t)=u_{left}(x,t),$$

$$u(x,t)=u_{right}(x,t)$$

respectively.

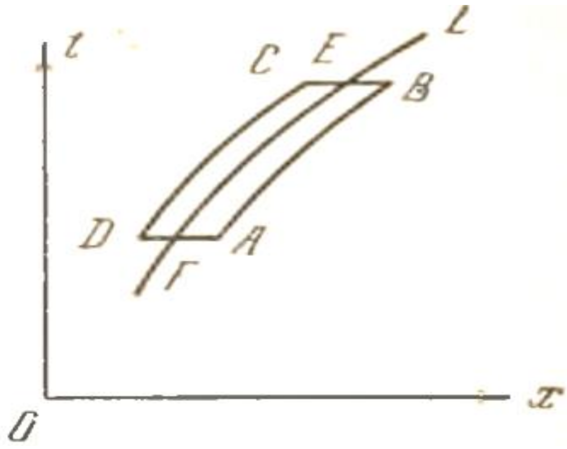


Figure 5.

It is known that the functions $u_{left}(x,t)$, $u_{right}(x,t)$, and the breaking point $\dot{x}=\frac{dx}{dt}$ cannot be arbitrary, they are connected by some relation.

Suppose that L is a dashed line (Figure 5). The integral on the contour ABCDA

$$\int_{ABCDA} u dx-\frac{u^2}{2}dt,$$

is the same as the integral on any other contour. If the intersections BC and DA converge at points E and F, respectively, the integral becomes zero and the following equation is formed:

$$\int_{L'}[u]dx-\left[\frac{u^2}{2}\right]dt=0, \text{ or } \int_{L'}\left([u]\frac{dx}{dt}-\left[\frac{u^2}{2}\right]\right)dt=0,$$

where $[z]=z_{right}-z_{left}$ is the jump of the quantity on the break line, and L' is the arbitrary part of the line $L'=EF$.

Since the part L' is arbitrary, the function under the integral must be zero at each point on the line L.

$$[u]\frac{dx}{dt}-\left[\frac{u^2}{2}\right]=0$$

from this we obtain equation

$$\frac{dx}{dt}=\left[\frac{u^2}{2}\right]:[u]=\frac{u_{right}^2-u_{left}^2}{2(u_{right}-u_{left})}=\frac{u_{left}-u_{right}}{2},$$

in which condition

$$\frac{dx}{dt}=\frac{u_{left}+u_{right}}{2} \tag{6}$$

is satisfied on the break line.

If we write equation $u_t+uu_x=0$ in another divergent form,

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^3}{3}\right)=0\cdot \tag{7}$$

In the same way as described above, we come to another integral relationship:

$$\oint_G \frac{u^2}{2}dx - \frac{u^3}{3}dt = 0. \qquad (8)$$

At the break, we have another condition:

$$\frac{dx}{dt} = \frac{2}{3}\frac{u_{left}^2 + u_{left}u_{right} + u_{right}^2}{u_{left} + u_{right}} \qquad (9)$$

The slope of the break line (or shock wave velocity $\dot{x}$) does not coincide with the slope (6) of the first divergent record (4). It follows that the concept of a generalized solution depends on exactly what integral conservation law reflects the given differential equation (1). In mathematical physics, the laws of integral conservation have a specific physical meaning. For smooth u solutions, the following five forms of integral conservation laws are equally strong:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) = 0,$$

$$\oint_G u\,dx - \frac{u^2}{2}dt = 0,$$

$$\oint_G \frac{u^2}{2}dx - \frac{u^3}{3}dt = 0.$$

In the following consideration of the Cauchy problem (1), we assume that the integral conservation law (5) and the condition (6) obtained from it for the break line are satisfied.

Here is a different definition of a generalized solution. The following ancillary problem can be considered in determining the generalized solution of the differential problem (1):

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \\ &u(x,0) = \psi(x) \end{aligned}\right\} \qquad (10)$$

where differential equation (1) is no longer hyperbolic, but parabolic. Its solution maintains smoothness if $\psi(x)$ is a smooth function. If $u(x,0) = \psi(x)$ has a break, the break is smoothed. The parameter $\mu > 0$ acts as a viscosity in the dynamics of gases. The limit of the solution of the problem (10) at $\mu \to 0$ can be considered as the generalized solution of the problem (1).

This definition of a generalized solution is consistent with the law of conservation (5).

Artificial adhesive scheme. Let us consider the problem of constructing a differential scheme

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \\ &u(x,0) = \psi(x) \end{aligned}\right\} \qquad (11)$$

corresponding to this differential problem. For clarity, we take $\psi(x) > 0$. In this case $u(x,t) > 0$. At first glance, it seems natural to use the following scheme:

$$\left.\begin{aligned} \frac{u_m^{p+1} - u_m^p}{\tau} + u_m^p \frac{u_m^p - u_{m-1}^p}{h} = 0,\ p = 0,1,\ldots, m = 0,\pm 1,\ldots, \\ u_m^0 = \psi(x_m) \end{aligned}\right\} \quad (12)$$

If we fix the coefficient u_m^p at the point $m = m_0$, the coefficients in the resulting equation are constant, and the principle of maximality holds in the transition in layer $t = (p+1)\tau$, if the pitch of the grid $t = \tau_p$ is chosen to satisfy the following condition $r_p = \frac{\tau_p}{h} \le \frac{1}{\max_m |u_m^p|}$.

So, bottom line is that we're really looking forward to it. If the solution of problem (1) is smooth, then there is no doubt that the problem (1) is approximated by the scheme (2). Indeed, in this case, experimental calculations performed on previously known smooth solutions confirm the approximation of the scheme. However, if the problem (1) has a discontinuous solution, then we cannot expect to approach the generalized solution given by the law of integral conservation

$$\oint_G u dx - \frac{u^2}{2} dt = 0. \quad (13)$$

Therefore, the integral conservation laws corresponding to the generalized solution sought, such as law (3) or equation with artificial adhesion (10), should be used to construct the differential schemes:

$$u_t + uu_x = \mu u_{xx}. \quad (14)$$

This equation allows us to choose the generalized solution of interest at $\mu \to 0$. To do this, let's look at a circuit with artificial adhesives. It should be noted that the differential scheme

$$\frac{u_m^{p+1} - u_m^p}{\tau} + u_m^p \frac{u_m^p - u_{m-1}^p}{h} = \mu \frac{u_{m-1}^p - 2u_m^p + u_{m+1}^p}{h^2},$$

$$u_m^0 = \psi(x_m)$$

which includes an artificial small viscosity, has this $u^{(h)} = \{u_m^p\}$ solution, which is small enough for $h \to 0$ and at a sufficiently small $\tau = \tau(h,\mu)$ for the generalized solution of the problem (1) approaching flat around. In this case, the amount $\mu = \mu(h)$ must be slow enough to zero at $h \to 0$. Circuits with artificial viscosity have been successfully applied to gas dynamics equations. Their main drawback is that they close interruptions. There are two ways to handle this. In the first method, it is necessary to use not only the law of integral conservation (3), but also the condition of termination

$$\frac{dx}{dt} = \frac{u_{left} + u_{right}}{2} \quad (15)$$

arising from it.

In the second approach, the interruption is not separated and the calculations are performed on the basis of uniform formulas at all compute nodes.

For this

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2},\\&u(x,0)=\psi(x)\end{aligned}\right\}$$

Koshi issue, consider the following:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0.001\frac{\partial^2 u}{\partial x^2},\\&u(x,0)=x^2+2x+1\end{aligned}\right\}$$

this problem is solved by the following algorithm:

$$\left.\begin{aligned}&u_m^{p+1}=u_m^p-\tau u_m^p\frac{u_m^p-u_{m-1}^p}{h}+\tau\mu\frac{u_{m-1}^p-2u_m^p+u_{m+1}^p}{h^2},\ p=0,1,\ldots,M\\&u(x,0)=u_m^0=\psi(x_m)=x_m^2+2x_m+1,\ m=0,\pm1,\pm2,\ldots,\pm N\end{aligned}\right\}$$

where $x_m=m\cdot h$, $t_p=p\cdot\tau$, $h=\frac{1}{N}$, $\tau=\frac{T}{M}$, $0\le\mu<<1$ must satisfy the condition, so that $\mu=0.001$ is taken in the given matter. The exact solution is calculated by the function $u(x_m,t_p)=x_m^2+2x_m+t_p+1$. We perform calculations for the given problem $N=10$, $M=3$, $T=0.03$ values.

The exact solution is the values of y[i][j]

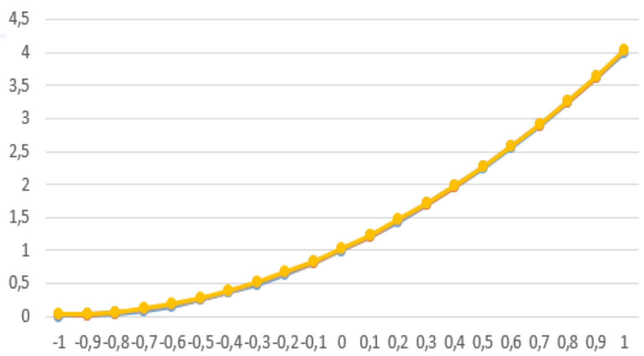
y[-10][0] = 0	y[-10][1] = 0,01	y[-10][2] = 0,02	y[-10][3] = 0,03
y[-9][0] = 0,01	y[-9][1] = 0,02	y[-9][2] = 0,03	y[-9][3] = 0,04
y[-8][0] = 0,04	y[-8][1] = 0,05	y[-8][2] = 0,06	y[-8][3] = 0,07
y[-7][0] = 0,09	y[-7][1] = 0,1	y[-7][2] = 0,11	y[-7][3] = 0,12
y[-6][0] = 0,16	y[-6][1] = 0,17	y[-6][2] = 0,18	y[-6][3] = 0,19
y[-5][0] = 0,25	y[-5][1] = 0,26	y[-5][2] = 0,27	y[-5][3] = 0,28
y[-4][0] = 0,36	y[-4][1] = 0,37	y[-4][2] = 0,38	y[-4][3] = 0,39
y[-3][0] = 0,49	y[-3][1] = 0,5	y[-3][2] = 0,51	y[-3][3] = 0,52
y[-2][0] = 0,64	y[-2][1] = 0,65	y[-2][2] = 0,66	y[-2][3] = 0,67
y[-1][0] = 0,81	y[-1][1] = 0,82	y[-1][2] = 0,83	y[-1][3] = 0,84
y[0][0] = 1	y[0][1] = 1,01	y[0][2] = 1,02	y[0][3] = 1,03
y[1][0] = 1,21	y[1][1] = 1,22	y[1][2] = 1,23	y[1][3] = 1,24
y[2][0] = 1,44	y[2][1] = 1,45	y[2][2] = 1,46	y[2][3] = 1,47
y[3][0] = 1,69	y[3][1] = 1,7	y[3][2] = 1,71	y[3][3] = 1,72
y[4][0] = 1,96	y[4][1] = 1,97	y[4][2] = 1,98	y[4][3] = 1,99
y[5][0] = 2,25	y[5][1] = 2,26	y[5][2] = 2,27	y[5][3] = 2,28
y[6][0] = 2,56	y[6][1] = 2,57	y[6][2] = 2,58	y[6][3] = 2,59
y[7][0] = 2,89	y[7][1] = 2,9	y[7][2] = 2,91	y[7][3] = 2,92
y[8][0] = 3,24	y[8][1] = 3,25	y[8][2] = 3,26	y[8][3] = 3,27
y[9][0] = 3,61	y[9][1] = 3,62	y[9][2] = 3,63	y[9][3] = 3,64
y[10][0] = 4	y[10][1] = 4,01	y[10][2] = 4,02	y[10][3] = 4,03

The solution of the artificial viscous circuit is the values of u[m][p+1]

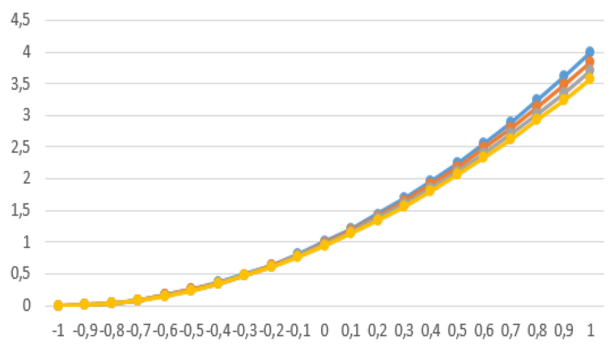
u[-10][0] = 0	u[-10][1] = 0,00002	u[-10][2] = 0,00004	u[-10][3] = 0,00006
u[-9][0] = 0,01	u[-9][1] = 0,01001	u[-9][2] = 0,0100199	u[-9][3] = 0,0100297
u[-8][0] = 0,04	u[-8][1] = 0,0399	u[-8][2] = 0,0398005	u[-8][3] = 0,0397015
u[-7][0] = 0,09	u[-7][1] = 0,08957	u[-7][2] = 0,0891448	u[-7][3] = 0,0887242
u[-6][0] = 0,16	u[-6][1] = 0,1589	u[-6][2] = 0,1578179	u[-6][3] = 0,1567532
u[-5][0] = 0,25	u[-5][1] = 0,24777	u[-5][2] = 0,2455875	u[-5][3] = 0,2434508
u[-4][0] = 0,36	u[-4][1] = 0,35606	u[-4][2] = 0,3522235	u[-4][3] = 0,3484862
u[-3][0] = 0,49	u[-3][1] = 0,48365	u[-3][2] = 0,4774983	u[-3][3] = 0,4715348
u[-2][0] = 0,64	u[-2][1] = 0,63042	u[-2][2] = 0,6211864	u[-2][3] = 0,6122789
u[-1][0] = 0,81	u[-1][1] = 0,79625	u[-1][2] = 0,7830647	u[-1][3] = 0,7704066
u[0][0] = 1	u[0][1] = 0,98102	u[0][2] = 0,9629125	u[0][3] = 0,9456125
u[1][0] = 1,21	u[1][1] = 1,18461	u[1][2] = 1,160511	u[1][3] = 1,137597
u[2][0] = 1,44	u[2][1] = 1,4069	u[2][2] = 1,375645	u[2][3] = 1,346067
u[3][0] = 1,69	u[3][1] = 1,64777	u[3][2] = 1,608099	u[3][3] = 1,570735
u[4][0] = 1,96	u[4][1] = 1,9071	u[4][2] = 1,857662	u[4][3] = 1,811318
u[5][0] = 2,25	u[5][1] = 2,18477	u[5][2] = 2,124124	u[5][3] = 2,067541
u[6][0] = 2,56	u[6][1] = 2,48066	u[6][2] = 2,407278	u[6][3] = 2,339131
u[7][0] = 2,89	u[7][1] = 2,79465	u[7][2] = 2,706919	u[7][3] = 2,625825
u[8][0] = 3,24	u[8][1] = 3,12662	u[8][2] = 3,022843	u[8][3] = 2,92736
u[9][0] = 3,61	u[9][1] = 3,47645	u[9][2] = 3,354847	u[9][3] = 3,243474
u[10][0] = 4	u[10][1] = 3,83961	u[10][2] = 3,695968	u[10][3] = 3,565854

Function graphs

Exact solution graph



Generalized solution graph



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