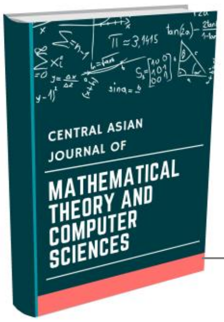

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On the Structural Properties of the Set of Controllability for Differential Inclusion Under Condition Mobility of Terminal Set

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Annotation:

In the paper we consider a mathematical model of a dynamic control system in the form of a differential inclusion. The property of controllability of this system under conditions mobility of terminal set M is researched. For this model of dynamical system the structural properties of the set of M-controllability are studied.

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1. INTRODUCTION

In the modern theory of dynamic control systems, differential inclusions (differential equations with a multi-valued right-hand side) are used as a convenient mathematical tool for research [1–5]. We can say that differential inclusions act as a mathematical model for many applied problems. They have broad applications to differential games, control problems under conditions of information inaccuracy and parameter uncertainty, problems of mathematical economics, and other mathematical issues. And this has led to an increase in interest in such models and the expansion of the scope of their research. Differential inclusions with delays, integro-differential inclusions, impulsed differential inclusions, differential inclusions with a fuzzy right-hand side, controlled differential inclusions, and others are studied [6–14].

The theory of differential inclusions has its own specific questions, such as the closeness, compactness, convexity and connectivity of the solution set, the properties of the integral funnel and the reachability set, and others [1,3,4]. They are important in the study of the optimization problem for functional defined on the trajectories of differential inclusions[1,5,6,7]. In the research of various problems of the theory of differential inclusions, multivalued maps, methods of convex and nonsmooth analysis are widely used [2–4,7].

Usually, in every optimization problem for a dynamic control system, it is assumed that the system has the property of controllability, i.e., it is possible to achieve the desired terminal (final) state with the help of controlled movements–trajectories emerging from a set of initial states. Quite a lot of works are devoted to this issue. For certain classes of stationary and non-stationary systems, necessary and sufficient conditions of controllability are found [9,10,13,14].

For dynamical systems, one of the actual problems is the study of the controllability property of permissible trajectories of differential inclusion with respect to given terminal states [1,3,4]. The controllability property of a differential inclusion takes on a special meaning in cases where the terminal state is mobile (time-dependent) or it is required to reach a mobile terminal set.

2. STATEMENT OF THE PROBLEM. RESEARCH METHODS.

Consider a mathematical model of a dynamical system in the form of a differential inclusion

$$\frac{dx}{dt} \in F(t,x). \quad (1)$$

By the permissible trajectories of the system under consideration, we will understand every absolutely continuous *n*-vector -function $x = x(t), t \in T = [t_0, t_1]$ that satisfies almost everywhere on $T = [t_0, t_1]$ a given differential inclusion.

Definition 1. The controllability set of the differential inclusion (1) relatively of the terminal state $x_1 \in R^n$ is the set of all those points $x_0 \in R^n$ for which there is a valid trajectory $x(t) = x(t, x_0)$, $T = [t_0, t_1]$, such that it reaches the terminal state: $x(t_0) = x_0, x(t_1) = x_1$.

In particular, when $x_1 = 0$ we obtain the concept of a zero-controllability set [14], i.e., all those points $x_0 \in R^n$ from which the origin of coordinates along the trajectories of the differential inclusion ($x(t_0) = x_0$, $x(t_1) = 0$) is achievable .

Let us now generalize the above notion of a controllability set, assuming that a mobile, i.e. time-dependent terminal set $M = M(t), t \geq t_0$ is given. In particular, the terminal set can be single-point: $M(t) = \{m(t)\}$ or constant: $M(t) \equiv M, t \geq t_0$.

Definition 2. The controllability set of the differential inclusion (1) relatively of the terminal set $M = M(t)$ (or briefly, the M -controllability set M) is the set of all such points $x_0 \in R^n$, $x_0 \notin M(t_0)$ for which there is a valid trajectory $x(t) = x(t, x_0)$, $T = [t_0, t_1]$ such that $x(t_0) = x_0$, $x(t_1) \in M(t_1)$.

Let us denote by $W(M, F)$ the set of $M-$controllability of the differential inclusion (1). Let $X(t_0, t_1, x_0, F)$ be the set of reachability of the differential inclusion (1) from the starting point $x_0 \in R^n$ at time $t_1 > t_0$, i.e., the set of all possible points $x_1 \in R^n$ for which there are trajectories $x = x(t)$, $t \in T = [t_0, t_1]$, such that $x(t_0) = x_0$ and $x(t_1) = x_1$.The set of reachability $X(t_0, t_1, x_0, F)$ is the section of the integral funnel $H(t_0, x_0) = \{(t, x) : t \geq t_0, x = x(t)\}$ of the differential inclusion (1) at time $t = t_1$.

From definition 2, it is clear that point $x_0 \in R^n$ is a $M-$controllability point of a differential inclusion (1) if and only if there exists $t_1 > t_0$ such that $X(t_0, t_1, x_0, F) \cap M(t_1) \neq \varnothing$, where $x_0 \notin M(t_0)$. Thus, the structural properties of the $M-$controllability set of the differential inclusion (1) can be studied by studying the structure of the set of the form

$$K(t_0, t_1, M, F) = \left\{\xi \in R^n : X_T(t_0, t_1, \xi, F) \cap M(t_1) \neq \varnothing\right\}, t_1 > t_0.$$

And the latter depends on properties $M = M(t)$ and $F = F(t, x)$.

From the definition of sets $W(M,F)$ and $K(t_0,t_1,M,F)$, the validity of the following equality easily follows

$$W(M,F)=(\bigcup_{t_1>t_0} K(t_0,t_1,M,F))\setminus M(t_0).$$

Clearly, if $F_1(t,x)\subset F_2(t,x)$, then $X(t_0,t_1,\xi,F_1)\subset X(t_0,t_1,\xi,F_2)$. Therefore, if $M_1(t)\subset M_2(t),\ t\ge t_0$, then $X(t_0,t_1,\xi,F_2)\cap M_2(t_1)\ne\varnothing$ follows from $X(t_0,t_1,\xi,F_1)\cap M_1(t_1)\ne\varnothing$. Therefore,

$$K(t_0,t_1,M_1,F_1)\subset K(t_0,t_1,M_2,F_2),\ W(M_1,F_1)\subset W(M_2,F_2).$$

Hence, in particular, we obtain that if there are maps $A:R^1\to R^{nxn},\ B:R^1\to\Omega(R^n),$ such that $A(t)x+B(t)\subset F(t,x)\ \forall(t,x)\in R^1\times R^n$, then the structure of the set $M-$controllability of the differential inclusion (1) will depend on such properties of the set $M-$controllability of the differential inclusion

$\dot{x}\in A(t)x+B(t)$. (2)

We will study the properties of the set $M-$controllability $W(M,A,B)$ of the differential inclusion (2). In the future, we will assume that the following conditions are met:

1. the elements of matrix $A(t)$ are measurable on any $T=[t_0,t_1]\subset[t_0,+\infty]$ and $\|A(t)\|\le a(t)$, where $a(\cdot)\in L_1(T)$;
2. the multi-valued map $t\to B(t)\in\Omega(R^n)$ is measurable on any segment $T=[t_0,t_1]\subset[t_0,+\infty]$ and $\|B(t)\|\le b(t)$, where $b(\cdot)\in L_1(T)$.

3. MAIN RESULTS.

Denoting by $X(t_0,t_1,\xi,A,B)$ the set of reachability of the differential inclusion (2), we define the set

$$K(t_0,t_1,M,A,B)=\{\xi\in R^n : X(t_0,t_1,\xi,A,B)\cap M(t_1)\ne\varnothing\}.$$

It is clear that the structural properties of the set $W(M,A,B)$ are expressed in terms of similar properties of sets of the form $K(t_0,t_1,M,A,B)$.

It is known [7] that for the reachability set $X(t_0,t_1,\xi,A,B)$, the formula is valid

$$X(t_0,t_1,\xi,A,B)=\Phi(t,t_0)\xi+\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)d\tau, \qquad (3)$$

where $\Phi(t,\tau)$ is the fundamental matrix of solutions to equation $\dot{x}=A(t)x, t\in T$. From this formula and the properties of the integral of multi-valued maps, it easily follows that the set $X(t_0,t_1,\xi,A,B)$ is a convex compact of R^n.

The relation of $X(t_0,t_1,\xi,A,B)\cap M(t_1)\ne\varnothing$ is equal to the inclusion of $0\in X(t_0,t_1,\xi,A,B)-M(t_1)$. Therefore

$$K(t_0,t_1,\xi,A,B)=\{\xi\in R^n : 0\in X(t_0,t_1,\xi,A,B)-M(t_1)\}.$$

Now, using the last equality and formula (3), we get the following result.

Theorem 1. The set $K(t_0,t_1,M,A,B)$ is represented by the formula

$$K(t_0,t_1,M,A,B)=-\int_{t_0}^{t_1}\Phi(t_0,t)B(t)dt+\Phi(t_0,t_1)M(t_1). \qquad (4)$$

Corollary 1. If $M(t_1)$ is a convex compact, then $K(t_0,t_1,M,A,B)$ is also a convex compact of R^n. If $M(t_1)$ and $convB(t)$ are strictly convex at $t\in T=[t_0,t_1]$, then $K(t_0,t_1,M,A,B)$ is strictly convex.

Let's say: $K_0(t_1,A,B)=K(t_0,t_1,\{0\},A,B)$. Then it is clear from formula (4) that

$$K_0(t_0,t_1,A,B)=-\int_{t_0}^{t_1}\Phi_A(t_0,t)B(t)dt. \qquad (5)$$

The set $K_0(t_0,t_1,A,B)$ is a convex compact of R^n. Taking into account the equality (5), the formula (4) takes the form:

$$K(t_0,t_1,M,A,B)=K_0(t_0,t_1,A,B)+\Phi_A(t_0,t_1)M(t_1). \qquad (6)$$

If $M(t_1)$ is a convex compact, then equality (6) can be written as the geometric difference:

$$K(t_0,t_1,M,A,B)\overset{*}{-}K_0(t_0,t_1,A,B)=\Phi_A(t_0,t_1)M(t_1).$$

Let $X_T^0(t_0,t_1,A,B)=X_T(t_0,t_1,0,A,B)$ be the reachability set of system (3) at $x_0=0$. Then we have:

Corollary 2. The formula is valid

$$K(t_0,t_1,M,A,B)=-\Phi_A(t_0,t_1)[X_T^0(t_0,t_1,A,B)+M(t_1)].$$

Theorem 2. Let $t_0<\bar{t}<t_1$. Then:

$$K(t_0,t_1,M,A,B)=K(t_0,\bar{t},\overline{M},A,B), \quad (7)$$

where $\overline{M}(t)=K(t,t_1,M,A,B)$.

In fact, using the formula (4), we have:

$$K(t_0,t_1,M,A,B)=-\int_{t_0}^{t_1}\Phi_A(t_0,t)B(t)dt+\Phi_A(t_0,t_1)M(t_1)=-\int_{t_0}^{\bar{t}}\Phi_A(t_0,t)B(t)dt+$$

$$+\Phi_A(t_0,\bar{t})[-\int_{\bar{t}}^{t_1}\Phi_A(\bar{t},t)B(t)dt+\Phi_A(\bar{t},t_1)M^0]=-\int_{t_0}^{\bar{t}}\Phi_A(t_0,t)B(t)dt+\Phi_A(t_0,\bar{t})K(\bar{t},t_1,M,A,B).$$ Now, assuming $\overline{M}(t)=K(t,t_1,M,A,B)$, from the last equality we get the formula (7).

Corollary 3. Let $t_0<\bar{t}<t_1$. Then the relation

$$K(t_0,t_1,M,A,B)\subset K(t_0,\bar{t},M,A,B) \quad (8)$$

holds if and only if $K(\bar{t},t_1,M,A,B)\subset M(\bar{t})$.

Theorem 3. Let $A(t)\equiv A,\ B(t)\equiv B$ be $t\in T=[t_0,t_1]$. Then:

$K(t_0,t_1,M,A,B)=\bigcup_{\xi\in M(t_1)} X(t_0,t_1,\xi,-A,-B)$. (9)

In fact, using the formula (3), we can write the following representation

$$\bigcup_{\xi\in M(t_1)} X(t_0,t_1,\xi,-A,-B)=\Phi(t_0,t_1)M(t_1)-\int_{t_0}^{t_1}\Phi(t,t_1)Bdt .$$

It is not difficult to see that $\Phi(t_1+t_0-s,t_1)=\Phi(t_0,s)$ is for all $s\in[t_0,t_1]$. Taking this into account and making the substitution of variables $s=t_1+t_0-t$ in the integral of the last equality, we get:

$$\bigcup_{\xi\in M(t_1)} X(t_0,t_1,\xi,-A,-B)=\Phi(t_0,t_1)M(t_1)-\int_{t_0}^{t_1}\Phi(t_0,s)Bds .$$

By virtue of Theorem 1, the right-hand side of the last equality is the set $K(t_0,t_1,M,A,B)$.

Let $0\in B(t)\ \forall t\in[t_0,t_1]$. Then:

$$r(t)=\inf_{\|\psi\|=1} C(B(t),\psi)\geq 0\ \forall t\in[t_0,t_1].$$

Suppose $r(t)>0$ for $t\in(t_0,t_0+\varepsilon)$, where $t_0+\varepsilon<t_1$. Then we have:

$$C(K(t_0,t_1,M,A,B),\psi)=C(K_0(t_0,t_1,M,A,B),\psi)+C(\Phi(t_0,t_1)M(t_1)),$$

$$C(K_0(t_0,t_1,M,A,B),\psi)=\int_{t_0}^{t_1}C(\Phi(t_0,t)B(t),-\psi)dt\geq\int_{t_0}^{t_1}C(\Phi(t_0,t)S_{r(t)},-\psi)dt\geq$$

$$=\int_{t_0}^{t_1}r(t)\|\Phi'(t_0,t)\psi\|dt\geq\int_{t_0}^{t_1}r(t)\inf_{\|\psi\|=1}\|\Phi'(t_0,t)\psi\|dt=L\|\psi\|>0 \text{ при } \|\psi\|=1,$$

$$C(K(t_0,t_1,M,A,B),\psi)\geq C(\Phi(t_0,t_1)M(t_1))+L\|\psi\|=C(\Phi(t_0,t_1)M(t_1)+S_L,\psi) \text{ при } \|\psi\|=1.$$

Therefore,

$$K(t_0,t_1,M,A,B)\supset\Phi(t_0,t_1)M(t_1)+S_L .$$

So, we got the following result.

Theorem 4. Let there be $\varepsilon>0$ such that $t_0+\varepsilon<t_1$ and $r(t)>0$ at $t\in(t_0,t_0+\varepsilon)$. Then the M-controllability set of system (2) contains some neighborhood of set $\Phi(t_0,t_1)M(t_1)$.

Corollary 4. Let $B(t)\equiv B$, $M(t)\equiv M$, $0\in \operatorname{int} B$, $0\in \operatorname{int} M$. Then the set $W(M,A,B)$ contains any neighborhood of point $\xi=0$.

4. DISCUSSION OF THE RESULTS AND CONCLUSION.

In this paper, the methods of research of works [7,10,11] are developed. From the results obtained, we note the representation formula (4) for the set $K(t_0,t_1,M,A,B)$. Using this formula, we have studied some properties of the set $K(t_0,t_1,M,A,B)$. The formulas (7) and (8) that follow from Theorem 1 give an idea of the dynamics of the set $K(t_0,t_1,M,A,B)$. In Theorem 3, formula (9) is given, which indicates the relationship of the sets of M-controllability with the set of reachability with the stationarity of the differential inclusion (2). Theorem 4 and its corollary generalize the results previously known for a linear dynamic control system [14].

In this paper, the problem of controllability of trajectories of differential inclusions is investigated for the case of mobility of a terminal set M. The studied properties of set $K(t_0, t_1, M, F)$ allow us to clarify the structure of the M-controllability set of the considered class of differential inclusions.

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