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An Introduction to Graph Theory and its Applications

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Annotation:
In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called links or lines). A distinction is made between undirected graphs, where edges link two vertices symmetrically, and directed graphs, where edges link two vertices asymmetrically. Graphs are one of the principal objects of study in discrete mathematics.

A R T I C L E I N F O

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INTRODUCTION

Graphs can be used to model many types of relations and processes in physical, biological,[7][8] social and information systems.[9] Many practical problems can be represented by graphs. Emphasizing their application to real-world systems, the term network is sometimes defined to mean a graph in which attributes (e.g. names) are associated with the vertices and edges, and the subject that expresses and understands real-world systems as a network is called network science.[1,2,3]

Within computer science, causal and non-causal linked structures are graphs that are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. For instance, the link structure of a website can be represented by a directed graph, in which the vertices represent web pages and directed edges represent links from one page to another. A similar approach can be taken to problems in social media,[10] travel, biology, computer chip design, mapping the progression of neuro-degenerative diseases,[11][12] and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. The transformation of graphs is often formalized and represented by graph rewrite systems. Complementary to graph transformation systems focusing on rule-based in-memory manipulation of graphs are graph databases geared towards transaction-safe, persistent storing and querying of graph-structured data.

Linguistics

Graph-theoretic methods, in various forms, have proven particularly useful in linguistics, since natural language often lends itself well to discrete structure. Traditionally, syntax and compositional semantics follow tree-based structures, whose expressive power lies in the principle of compositionality, modeled in a hierarchical graph. More contemporary approaches such as head-driven phrase structure grammar model the syntax of natural language using typed feature structures, which are directed acyclic graphs. Within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words; semantic networks are therefore important in computational linguistics. Still, other methods in phonology (e.g. optimality theory, which uses lattice graphs) and morphology (e.g. finite-state morphology, using finite-state transducers) are common in the analysis of language as a graph. Indeed, the usefulness of this area of mathematics to linguistics has borne organizations such as TextGraphs, as well as various 'Net' projects, such as WordNet, VerbNet, and others.

Physics and chemistry

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three-dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. Also, "the Feynman graphs and rules of calculation summarize quantum field theory in a form in close contact with the experimental numbers one wants to understand."[13] In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching.[4,5,6] In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems. Similarly, in computational neuroscience graphs can be used to represent functional connections between brain areas that interact to give rise to various cognitive processes, where the vertices represent different areas of the brain and the edges represent the connections between those areas. Graph theory plays an important role in electrical modeling of electrical networks, here, weights are associated with resistance of the wire segments to obtain electrical properties of network structures.[14] Graphs are also used to represent the micro-scale channels of porous media, in which the vertices represent the pores and the edges represent the smaller channels connecting the pores. Chemical graph theory uses the molecular graph as a means to model molecules. Graphs and networks are excellent models to study and understand phase transitions and critical phenomena. Removal of nodes or edges leads to a critical transition where the network breaks into small clusters which is studied as a phase transition. This breakdown is studied via percolation theory.[15]

Social sciences

Graph theory is also widely used in sociology as a way, for example, to measure actors' prestige or to explore rumor spreading, notably through the use of social network analysis software. Under the umbrella of social networks are many different types of graphs.[17] Acquaintanceship and friendship graphs describe whether people know each other. Influence graphs model whether certain people can influence the behavior of others. Finally, collaboration graphs model whether two people work together in a particular way, such as acting in a movie together.

Biology

Likewise, graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or inhabit) and the edges represent migration paths or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.[7,8,9]

Graphs are also commonly used in molecular biology and genomics to model and analyse datasets with complex relationships. For example, graph-based methods are often used to 'cluster' cells together into cell-types in single-cell transcriptome analysis. Another use is to model genes or proteins in a pathway and study the relationships between them, such as metabolic pathways and gene regulatory networks.[18] Evolutionary trees, ecological networks, and hierarchical clustering of gene expression patterns are also represented as graph structures.

Graph theory is also used in connectomics;[19] nervous systems can be seen as a graph, where the nodes are neurons and the edges are the connections between them.

Mathematics

In mathematics, graphs are useful in geometry and certain parts of topology such as knot theory. Algebraic graph theory has close links with group theory. Algebraic graph theory has been applied to many areas including dynamic systems and complexity.

Other topics

A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights, or weighted graphs, are used to represent structures in which pairwise connections have some numerical values. For example, if a graph represents a road network, the weights could represent the length of each road. There may be several weights associated with each edge, including distance (as in the previous example), travel time, or monetary cost. Such weighted graphs are commonly used to program GPS's, and travel-planning search engines that compare flight times and costs.

History

The paper written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736 is regarded as the first paper in the history of graph theory.[20] This paper, as well as the one written by Vandermonde on the knight problem, carried on with the analysis situs initiated by Leibniz. Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy[21] and L'Huilier,[22] and represents the beginning of the branch of mathematics known as topology.

More than one century after Euler's paper on the bridges of Königsberg and while Listing was introducing the concept of topology, Cayley was led by an interest in particular analytical forms arising from differential calculus to study a particular class of graphs, the trees.[23] This study had many implications for theoretical chemistry. The techniques he used mainly concern the enumeration of graphs with particular properties. [10,11,12]Enumerative graph theory then arose from the results of Cayley and the fundamental results published by Pólya between 1935 and 1937. These were generalized by De Bruijn in 1959. Cayley linked his results on trees with contemporary studies of chemical composition.[24] The fusion of ideas from mathematics with those from chemistry began what has become part of the standard terminology of graph theory.

In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams:[25] "[…] Every invariant and co-variant thus becomes expressible by a graph precisely identical with a Kekuléan diagram or chemicograph. […] I give a rule for the geometrical multiplication of graphs, i.e. for constructing a graph to the product of in- or co-variants whose separate graphs are given. […]" (italics as in the original).

The first textbook on graph theory was written by Dénes Kőnig, and published in 1936.[26] Another book by Frank Harary, published in 1969, was "considered the world over to be the definitive textbook on the subject",[27] and enabled mathematicians, chemists, electrical engineers and social scientists to talk to each other. Harary donated all of the royalties to fund the Pólya Prize.[28]

One of the most famous and stimulating problems in graph theory is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe, and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the factorization problems, particularly studied by Petersen and Kőnig. The works of Ramsey on colorations and more specially the results obtained by Turán in 1941 was at the origin of another branch of graph theory, extremal graph theory.[13,14,15]

The four color problem remained unsolved for more than a century. In 1969 Heinrich Heesch published a method for solving the problem using computers.[29] A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken makes fundamental use of the notion of "discharging" developed by Heesch.[30][31] The proof involved checking the properties of 1,936 configurations by computer, and was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given twenty years later by Robertson, Seymour, Sanders and Thomas.[32]

The autonomous development of topology from 1860 and 1930 fertilized graph theory back through the works of Jordan, Kuratowski and Whitney. Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use comes from the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's circuit laws for calculating the voltage and current in electric circuits.

The introduction of probabilistic methods in graph theory, especially in the study of Erdős and Rényi of the asymptotic probability of graph connectivity, gave rise to yet another branch, known as random graph theory, which has been a fruitful source of graph-theoretic results.

Representation

A graph is an abstraction of relationships that emerge in nature; hence, it cannot be coupled to a certain representation. The way it is represented depends on the degree of convenience such representation provides for a certain application. The most common representations are the visual, in which, usually, vertices are drawn and connected by edges, and the tabular, in which rows of a table provide information about the relationships between the vertices within the graph.

Visual: Graph drawing

Graphs are usually represented visually by drawing a point or circle for every vertex, and drawing a line between two vertices if they are connected by an edge. If the graph is directed, the direction is indicated by drawing an arrow. If the graph is weighted, the weight is added on the arrow.

A graph drawing should not be confused with the graph itself (the abstract, non-visual structure) as there are several ways to structure the graph drawing. All that matters is which vertices are connected to which others by how many edges and not the exact layout. In practice, it is often difficult to decide if two drawings represent the same graph. Depending on the problem domain some layouts may be better suited and easier to understand than others.

The pioneering work of W. T. Tutte was very influential on the subject of graph drawing. Among other achievements, he introduced the use of linear algebraic methods to obtain graph drawings.[16,17,18]

Graph drawing also can be said to encompass problems that deal with the crossing number and its various generalizations. The crossing number of a graph is the minimum number of intersections between edges that a drawing of the graph in the plane must contain. For a planar graph, the crossing number is zero by definition. Drawings on surfaces other than the plane are also studied.

There are other techniques to visualize a graph away from vertices and edges, including circle packings, intersection graph, and other visualizations of the adjacency matrix.

The tabular representation lends itself well to computational applications. There are different ways to store graphs in a computer system. The data structure used depends on both the graph structure and the algorithm used for manipulating the graph. Theoretically one can distinguish between list and matrix structures but in concrete applications the best structure is often a combination of both. List structures are often preferred for sparse graphs as they have smaller memory requirements. Matrix structures on the other hand provide faster access for some applications but can consume huge amounts of memory. Implementations of sparse matrix structures that are efficient on modern parallel computer architectures are an object of current investigation.[33]

List structures include the edge list, an array of pairs of vertices, and the adjacency list, which separately lists the neighbors of each vertex: Much like the edge list, each vertex has a list of which vertices it is adjacent to.[19,20,21]

Matrix structures include the incidence matrix, a matrix of 0's and 1's whose rows represent vertices and whose columns represent edges, and the adjacency matrix, in which both the rows and columns are indexed by vertices. In both cases a 1 indicates two adjacent objects and a 0 indicates two non-adjacent objects. The degree matrix indicates the degree of vertices. The Laplacian matrix is a modified form of the adjacency matrix that incorporates information about the degrees of the vertices, and is useful in some calculations such as Kirchhoff's theorem on the number of spanning trees of a graph. The distance matrix, like the adjacency matrix, has both its rows and columns indexed by vertices, but rather than containing a 0 or a 1 in each cell it contains the length of a shortest path between two vertices.

Problems

Enumeration

There is a large literature on graphical enumeration: the problem of counting graphs meeting specified conditions. Some of this work is found in Harary and Palmer (1973).

Subgraphs, induced subgraphs, and minors

A common problem, called the subgraph isomorphism problem, is finding a fixed graph as a subgraph in a given graph. One reason to be interested in such a question is that many graph properties are hereditary for subgraphs, which means that a graph has the property if and only if all subgraphs have it too. Unfortunately, finding maximal subgraphs of a certain kind is often an NP-complete problem. For example:

- Finding the largest complete subgraph is called the clique problem (NP-complete).

One special case of subgraph isomorphism is the graph isomorphism problem. It asks whether two graphs are isomorphic. It is not known whether this problem is NP-complete, nor whether it can be solved in polynomial time.

A similar problem is finding induced subgraphs in a given graph. Again, some important graph properties are hereditary with respect to induced subgraphs, which means that a graph has a property if and only if all induced subgraphs also have it. Finding maximal induced subgraphs of a certain kind is also often NP-complete. For example:

- Finding the largest edgeless induced subgraph or independent set is called the independent set problem (NP-complete).

Still another such problem, the minor containment problem, is to find a fixed graph as a minor of a given graph. A minor or subcontraction of a graph is any graph obtained by taking a subgraph and contracting some (or no) edges. Many graph properties are hereditary for minors, which means that a graph has a property if and only if all minors have it too. For example, Wagner's Theorem states:

- A graph is planar if it contains as a minor neither the complete bipartite graph $K_{3,3}$ (see the Three-cottage problem) nor the complete graph K_5.

A similar problem, the subdivision containment problem, is to find a fixed graph as a subdivision of a given graph. A subdivision or homeomorphism of a graph is any graph obtained by subdividing some (or no) edges. Subdivision containment is related to graph properties such as planarity. For example, Kuratowski's Theorem states:

- A graph is planar if it contains as a subdivision neither the complete bipartite graph $K_{3,3}$ nor the complete graph K_5.[22,23,24]

Another problem in subdivision containment is the Kelmans–Seymour conjecture:

- Every 5-vertex-connected graph that is not planar contains a subdivision of the 5-vertex complete graph K_5.

Another class of problems has to do with the extent to which various species and generalizations of graphs are determined by their point-deleted subgraphs. For example:

- The reconstruction conjecture

Graph coloring

Many problems and theorems in graph theory have to do with various ways of coloring graphs. Typically, one is interested in coloring a graph so that no two adjacent vertices have the same color, or with other similar restrictions. One may also consider coloring edges (possibly so that no two coincident edges are the same color), or other variations. Among the famous results and conjectures concerning graph coloring are the following:

- ✓ Four-color theorem
- ✓ Strong perfect graph theorem
- ✓ Erdős–Faber–Lovász conjecture
- ✓ Total coloring conjecture, also called Behzad's conjecture (unsolved)
- ✓ List coloring conjecture (unsolved)
- ✓ Hadwiger conjecture (graph theory) (unsolved)

Subsumption and unification

Constraint modeling theories concern families of directed graphs related by a partial order. In these applications, graphs are ordered by specificity, meaning that more constrained graphs—which are more specific and thus contain a greater amount of information—are subsumed by those that are more general. Operations between graphs include evaluating the direction of a subsumption relationship between two graphs, if any, and computing graph unification. The unification of two argument graphs is defined as the most general graph (or the computation thereof) that is consistent with (i.e. contains all of the information in) the inputs, if such a graph exists; efficient unification algorithms are known.

For constraint frameworks which are strictly compositional, graph unification is the sufficient satisfiability and combination function. Well-known applications include automatic theorem proving and modeling the elaboration of linguistic structure.[25,26,27]

Route problems

- ✓ Hamiltonian path problem
- ✓ Minimum spanning tree
- ✓ Route inspection problem (also called the "Chinese postman problem")
- ✓ Seven bridges of Königsberg

- ✓ Shortest path problem
- ✓ Steiner tree
- ✓ Three-cottage problem
- ✓ Traveling salesman problem (NP-hard)

Network flow

There are numerous problems arising especially from applications that have to do with various notions of flows in networks, for example:

- ✓ Max flow min cut theorem

Visibility problems

- ✓ Museum guard problem

Covering problems

Covering problems in graphs may refer to various set cover problems on subsets of vertices/subgraphs. [28, 29, 30]

- ✓ Dominating set problem is the special case of set cover problem where sets are the closed neighborhoods
- ✓ Vertex cover problem is the special case of set cover problem where sets to cover are every edges.
- ✓ The original set cover problem, also called hitting set, can be described as a vertex cover in a hypergraph.

Decomposition problems

Decomposition, defined as partitioning the edge set of a graph (with as many vertices as necessary accompanying the edges of each part of the partition), has a wide variety of questions. Often, the problem is to decompose a graph into subgraphs isomorphic to a fixed graph; for instance, decomposing a complete graph into Hamiltonian cycles. Other problems specify a family of graphs into which a given graph should be decomposed, for instance, a family of cycles, or decomposing a complete graph K_n into $n - 1$ specified trees having, respectively, 1, 2, 3, ..., $n - 1$ edges.

Some specific decomposition problems that have been studied include:

- ✓ Arboricity, a decomposition into as few forests as possible
- ✓ Cycle double cover, a decomposition into a collection of cycles covering each edge exactly twice
- ✓ Edge coloring, a decomposition into as few matchings as possible
- ✓ Graph factorization, a decomposition of a regular graph into regular subgraphs of given degrees

Graph classes

Many problems involve characterizing the members of various classes of graphs. Some examples of such questions are below:

- ✓ Enumerating the members of a class
- ✓ Characterizing a class in terms of forbidden substructures
- ✓ Ascertaining relationships among classes (e.g. does one property of graphs imply another)
- ✓ Finding efficient algorithms to decide membership in a class
- ✓ Finding representations for members of a class

DISCUSSION

- Graph theory has many aspects. Graphs can be directed or undirected. An example of a directed graph would be the system of roads in a city. Some streets in the city are one way streets. This means, that on those parts there is only one direction to follow.
- Graphs can be weighted. An example would be a road network, with distances, or with tolls (for roads).
- The nodes (the circles in the schematic) of a graph are called vertices. The lines connecting the nodes are called edges. There can be no line between two nodes, there can be one line, or there can be multiple lines.
- In graph theory, Trees structures are widely used, they represent hierarchical structures. A Tree is a directed or undirected graph where there is no cycle, meaning: no way of going from one vertex (for example a town) to the same one using each edge you use only once (walking only once on each road you take).

A graph is an abstract data structure. It holds nodes that are usually related to each other. A node is a dataset, typically in the form of ordered pairs. Nodes are either connected or not connected to another node. The relation between nodes is usually defined as an Edge. Graphs are useful for their ability to associate nodes with other nodes. [31,32,33]There are a few representations of Graphs in practice.

Leonhard Euler used to live in a town called Königsberg. (Its name changed to Kaliningrad in 1946). The town is on the river Pregel. There is an island in the river. There are some bridges across the river. Euler wanted to walk around and use each of the bridges once. He asked if he could do this. In 1736, he published a scientific article where he showed that this was not possible. Today, this problem is known as the Seven Bridges of Königsberg. The article is seen as the first paper in the history of graph theory.[1]

This article, as well as the one written by Vandermonde on the knight problem, carried on with the analysis situs initiated by Leibniz. Euler's formula was about the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy[2] and L'Huillier,[3] and is at the origin of topology.

The fusion of the ideas coming from mathematics with those coming from chemistry is at the origin of a part of the standard terminology of graph theory. In particular, the term "graph" was introduced by Sylvester in an article published in 1878 in Nature.[4]

One of the most famous and productive problems of graph theory is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?"

Graph theory in perspective

Graph theory is an important part of mathematics and computer science. To many such problems, exact solutions do exist.[30,31] Many times however, they are very hard to calculate. Therefore, very often, approximations are used. There are two kinds of such approximations, Monte-Carlo algorithms and Las-Vegas algorithms.

RESULTS

In discrete mathematics, and more specifically in graph theory, a graph is a structure amounting to a set of objects in which some pairs of the objects are in some sense "related". The objects correspond to mathematical abstractions called vertices (also called nodes or points) and each of the related pairs of vertices is called an edge (also called link or line).[1] Typically, a graph is depicted in diagrammatic form as a set of dots or circles for the vertices, joined by lines or curves for the edges. Graphs are one of the objects of study in discrete mathematics.

The edges may be directed or undirected. For example, if the vertices represent people at a party, and there is an edge between two people if they shake hands, then this graph is undirected because any person A can shake hands with a person B only if B also shakes hands with A. In contrast, if an edge from a person A to a person B means that A owes money to B, then this graph is directed, because owing money is not necessarily reciprocated.

Graphs are the basic subject studied by graph theory. The word "graph" was first used in this sense by J. J. Sylvester in 1878 due to a direct relation between mathematics and chemical structure (what he called a chemico-graphical image).[2][3]

A graph (sometimes called an undirected graph to distinguish it from a directed graph, or a simple graph to distinguish it from a multigraph)[4][5] is a pair G = (V, E), where V is a set whose elements are called vertices (singular: vertex), and E is a set of paired vertices, whose elements are called edges (sometimes links or lines).

The vertices x and y of an edge {x, y} are called the endpoints of the edge. The edge is said to join x and y and to be incident on x and y. A vertex may belong to no edge, in which case it is not joined to any other vertex.

A multigraph is a generalization that allows multiple edges to have the same pair of endpoints. In some texts, multigraphs are simply called graphs.[6][7]

Sometimes, graphs are allowed to contain loops, which are edges that join a vertex to itself. To allow loops, the pairs of vertices in E must be allowed to have the same node twice. Such generalized graphs are called graphs with loops or simply graphs when it is clear from the context that loops are allowed.

Generally, the set of vertices V is supposed to be finite; this implies that the set of edges is also finite. Infinite graphs are sometimes considered, but are more often viewed as a special kind of binary relation, as most results on finite graphs do not extend to the infinite case, or need a rather different proof.

An empty graph is a graph that has an empty set of vertices (and thus an empty set of edges). The order of a graph is its number of vertices |V|. The size of a graph is its number of edges |E|. However, in some contexts, such as for expressing the computational complexity of algorithms, the size is |V| + |E| (otherwise, a non-empty graph could have size 0). The degree or valency of a vertex is the number of edges that are incident to it; for graphs [1]with loops, a loop is counted twice.

In a graph of order n, the maximum degree of each vertex is $n - 1$ (or $n + 1$ if loops are allowed, because a loop contributes 2 to the degree), and the maximum number of edges is $n(n - 1)/2$ (or $n(n + 1)/2$ if loops are allowed).

The edges of a graph define a symmetric relation on the vertices, called the adjacency relation. Specifically, two vertices x and y are adjacent if {x, y} is an edge. A graph may be fully specified by its adjacency matrix A, which is an $n \times n$ square matrix, with A_{ij} specifying the number of connections from vertex i to vertex j. For a simple graph, A_{ij} is either 0, indicating disconnection, or 1, indicating connection; moreover $A_{ii} = 0$ because an edge in a simple graph cannot start and end at the same vertex. Graphs with self-loops will be characterized by some or all A_{ii} being equal to a positive integer, and multigraphs (with multiple edges between vertices) will be characterized by some or all A_{ij} being equal to a positive integer. Undirected graphs will have a symmetric adjacency matrix (meaning $A_{ij} = A_{ji}$).

Directed graph

A directed graph or digraph is a graph in which edges have orientations.

In one restricted but very common sense of the term,[8] a directed graph is a pair G = (V, E) comprising:

- V, a set of vertices (also called nodes or points);
- E, a set of edges (also called directed edges, directed links, directed lines, arrows, or arcs), which are ordered pairs of distinct vertices:

To avoid ambiguity, this type of object may be called precisely a directed simple graph.

In the edge (x, y) directed from x to y, the vertices x and y are called the endpoints of the edge, x the tail of the edge and y the head of the edge. The edge is said to join x and y and to be incident on x and on y. A vertex may exist in a graph and not belong to an edge. The edge (y, x) is called the inverted edge of (x, y). Multiple edges, not allowed under the definition above, are two or more edges with both the same tail and the same head.

In one more general sense of the term allowing multiple edges,[8] a directed graph is an ordered triple G = (V, E, ϕ) comprising:

- V, a set of vertices (also called nodes or points);
- E, a set of edges (also called directed edges, directed links, directed lines, arrows or arcs);
- ϕ, an incidence function mapping every edge to an ordered pair of vertices (that is, an edge is associated with two distinct vertices):

To avoid ambiguity, this type of object may be called precisely a directed multigraph.

Mixed graph

A mixed graph is a graph in which some edges may be directed and some may be undirected. It is an ordered triple G = (V, E, A) for a mixed simple graph and G = (V, E, A, ϕ_E, ϕ_A) for a mixed multigraph with V, E (the undirected edges), A (the directed edges), ϕ_E and ϕ_A defined as above. Directed and undirected graphs are special cases.

Weighted graph

A weighted graph or a network[9][10] is a graph in which a number (the weight) is assigned to each edge.[11] Such weights might represent for example costs, lengths or capacities, depending on the problem at hand. Such graphs arise in many contexts, for example in shortest path problems such as the traveling salesman problem.

Types of graphs

Oriented graph

One definition of an oriented graph is that it is a directed graph in which at most one of (x, y) and (y, x) may be edges of the graph. That is, it is a directed graph that can be formed as an orientation of an undirected (simple) graph.

Some authors use "oriented graph" to mean the same as "directed graph". Some authors use "oriented graph" to mean any orientation of a given undirected graph or multigraph.

Regular graph

A regular graph is a graph in which each vertex has the same number of neighbours, i.e., every vertex has the same degree. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k.

Complete graph

A complete graph is a graph in which each pair of vertices is joined by an edge. A complete graph contains all possible edges.

Finite graph

A finite graph is a graph in which the vertex set and the edge set are finite sets. Otherwise, it is called an infinite graph.

Most commonly in graph theory it is implied that the graphs discussed are finite. If the graphs are infinite, that is usually specifically stated.

Connected graph

In an undirected graph, an unordered pair of vertices {x, y} is called connected if a path leads from x to y. Otherwise, the unordered pair is called disconnected.

A connected graph is an undirected graph in which every unordered pair of vertices in the graph is connected. Otherwise, it is called a disconnected graph.

In a directed graph, an ordered pair of vertices (x, y) is called strongly connected if a directed path leads from x to y. Otherwise, the ordered pair is called weakly connected if an undirected path leads from x to y after replacing all of its directed edges with undirected edges. Otherwise, the ordered pair is called disconnected.

A strongly connected graph is a directed graph in which every ordered pair of vertices in the graph is strongly connected. Otherwise, it is called a weakly connected graph if every ordered pair of vertices in the graph is weakly connected. Otherwise it is called a disconnected graph.

A k-vertex-connected graph or k-edge-connected graph is a graph in which no set of $k - 1$ vertices (respectively, edges) exists that, when removed, disconnects the graph. A k-vertex-connected graph is often called simply a k-connected graph.

Bipartite graph

A bipartite graph is a simple graph in which the vertex set can be partitioned into two sets, W and X, so that no two vertices in W share a common edge and no two vertices in X share a common edge. Alternatively, it is a graph with a chromatic number of 2.

In a complete bipartite graph, the vertex set is the union of two disjoint sets, W and X, so that every vertex in W is adjacent to every vertex in X but there are no edges within W or X.

Path graph

A path graph or linear graph of order $n \geq 2$ is a graph in which the vertices can be listed in an order $v_1, v_2, \ldots, v_n$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n - 1$. Path graphs can be characterized as connected graphs in which the degree of all but two vertices is 2 and the degree of the two remaining vertices is 1. If a path graph occurs as a subgraph of another graph, it is a path in that graph.

Planar graph

A planar graph is a graph whose vertices and edges can be drawn in a plane such that no two of the edges intersect.[31,32]

Cycle graph

A cycle graph or circular graph of order $n \geq 3$ is a graph in which the vertices can be listed in an order $v_1, v_2, \ldots, v_n$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n - 1$, plus the edge $\{v_n, v_1\}$. Cycle graphs can be characterized as connected graphs in which the degree of all vertices is 2. If a cycle graph occurs as a subgraph of another graph, it is a cycle or circuit in that graph.

Tree

A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph.

A forest is an undirected graph in which any two vertices are connected by at most one path, or equivalently an acyclic undirected graph, or equivalently a disjoint union of trees.

Polytree

A polytree (or directed tree or oriented tree or singly connected network) is a directed acyclic graph (DAG) whose underlying undirected graph is a tree.

A polyforest (or directed forest or oriented forest) is a directed acyclic graph whose underlying undirected graph is a forest.

CONCLUSION

Graph operations

There are several operations that produce new graphs from initial ones, which might be classified into the following categories:

- ➢ unary operations, which create a new graph from an initial one, such as:
- ✓ edge contraction,
- ✓ line graph,
- ✓ dual graph,
- ✓ complement graph,
- ✓ graph rewriting;
- ➢ binary operations, which create a new graph from two initial ones, such as:
- ✓ disjoint union of graphs,
- ✓ cartesian product of graphs,
- ✓ tensor product of graphs,
- ✓ strong product of graphs,
- ✓ lexicographic product of graphs,
- ✓ series–parallel graphs.

Generalizations

In a hypergraph, an edge can join more than two vertices.

An undirected graph can be seen as a simplicial complex consisting of 1-simplices (the edges) and 0-simplices (the vertices). As such, complexes are generalizations of graphs since they allow for higher-dimensional simplices.

Every graph gives rise to a matroid.

In model theory, a graph is just a structure. But in that case, there is no limitation on the number of edges: it can be any cardinal number, see continuous graph.

In computational biology, power graph analysis introduces power graphs as an alternative representation of undirected graphs.

In geographic information systems, geometric networks are closely modeled after graphs, and borrow many concepts from graph theory to perform spatial analysis on road networks or utility grids.[33]

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