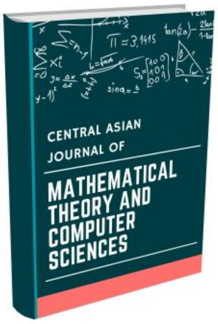

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Comparison of Two Methods for Linear Time Invariance Quadratic Optimal Control Problems

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ABSTRACT

In this paper we have studied the linear time invariance quadratic optimal control problems with unknown coefficients. The linear time invariance problems were parameterized based on control-state parameterization technique such that the objective function and the constraints are in terms of state variables and control variables. These two methods were converting the linear time invariance quadratic optimal control problems into quadratic programming problems and the converted problems were solved using MATLAB. When we increase the order of polynomial (M), then the computational results of the proposed methods gave better results but when we compare these two methods, Legendre scaling function was better than Chebyshev scaling function with regard to optimal value. Hence, the Legendre scaling function method is more suitable for solving the linear time invariance quadratic optimal control problems.

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1.INTRODUCTION

Optimal control is a science that deals with linear and nonlinear optimal control problems (OCPs) and the main objective of optimal control is to determine control function that will the cause of a systems (plant) to satisfy the physical constraints and at the same time extremize (maximize or minimize) a performance criterion (performance index or cost function) [1]. Analytical solutions of optimal control problems are not always available, thus finding an approximate solution is at least the most logical way to solve them. By contrast, linear optimal control problems have readily computable solutions rather than nonlinear optimal control problems. Optimal control problems with linear time varying systems are much more difficult to solve in comparison with linear time invariance systems [2].

The state variable (or function) is the set of variables (functions) used to describe the mathematical state of the system at a time t but according to [3] the value of state variable decreases when the time interval increases. The control variable (or function) is an operation that controls the recording, processing, or transmission of data at a time t but according to [6] the value of control variable increases when the time interval increases in the case of linear time invariant optimal control problems. The work of [5] presented the proof of multiplication of Legendre scaling function as well as the multiplication operational matrix which was used in the development of a computational method to the linear time-varying quadratic optimal control problems.

[4] Were solving both linear time invariance and linear time varying optimal control problems using Chebyshev wavelet. [6] Finding the approximate solution of the optimal control of linear time invariant systems using the Legendre wavelets and the operational matrix of stretch is derived and together with the operational matrix integration has been used to change the system of state equations into a set of algebraic equations.

2.METHODS

Using Legendre scaling function and Chebyshev scaling function convert the linear time invariance quadratic optimal control problems into quadratic programming problems. Then the converted quadratic programming problems were solved using MATLAB and sketch the graph of the approximate value of state and control variables. Then compare Legendre scaling function and Chebyshev scaling function with regard to optimal value.

3.CHEBYSHEV SCALING FUNCTION

We will introduce the definition, basis of Chebyshev scaling function. This basis will be the basic of this paper in the following section. Chebyshev scaling function is used to for approximate both state variables and control variables because the optimal solution of state variables and control variables are a function of time so it is better to use this function for quadratic optimal control problems and Chebyshev scaling function can be defined is as follows

$$\Psi_{nm}(t) = \begin{cases} \dfrac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} T_m(2^k t - 2n + 1), & \dfrac{n-1}{2^{k-1}} \le t \le \dfrac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$\text{where } \alpha_m = \begin{cases} \sqrt{2}, & m = 0 \\ 2, & m = 1,2,3,\dots \end{cases}$$

Here, $T_m(t)$ are the Chebyshev polynomials of order m,

$$T_0(t) = 1,$$
$$T_1(t) = t,$$
$$T_2(t) = 2t^2 - 1$$
$$T_3(t) = 4t^3 - 3t$$
$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \ m = 1, 2, 3, \dots . \quad (2)$$

3.1. Operational Matrix of Integration (OMI) for Chebyshev Scaling Function

Let P be an operational matrix of integration which is obtained from the integral of Chebyshev scaling function from 0 to t and these matrices play an important role to modeling the problems. According to [10], this matrix is used to change the system of state equations into a set of algebraic equations which can be solved using software. This is represented as follows:

$$\int_0^t \Psi(\tau)d\tau = P\Psi(t) \quad (3)$$

$$\Psi(t) = [\Psi_{1m}(t), \Psi_{2m}(t), \Psi_{3m}(t), \dots, \Psi_{2^{k-1}m}(t)]^T$$

So the matrix P is the cofficent of $\Psi(t)$

$$P = \begin{bmatrix} C & S \\ O & C \end{bmatrix}$$

where P is a $(2^{K-1}(M+1) \times 2^{K-1}(M+1))$ operational matrix of integration and O, C, S are $(M+1) \times (M+1)$ matrices and give

$$S = \frac{\sqrt{2}}{2^{K-1}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$C = \frac{1}{2^{k-1}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)} \\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{bmatrix}$$

O is matrix that all entire elements are zeros

3.2. Continuity Test of Chebyshev Scaling Function

To insure the continuity of the state variables between the different sections so we must add constraints. There are $2^{k-1} - 1$ points at which the continuity of the state variables has to ensure. These points are

$$t_i = \frac{i}{2^{k-1}} \quad , i = 1, 2, \dots, 2^{K-1} - 1$$

In addition, there is $(2^{K-1} - 1)$s equality constraints can be given as follows

$$(I_s \otimes \Phi')a = o_{(2^{K-1}-1)_{s\times 1}} \quad (4)$$

However, the dimension of continuity matrix Φ' or ensured constraints is $(2^{K-1} - 1) \times 2^{K-1}(M+1)$.

4. LEGENDRE SCALING FUNCTION

In this section, we will introduce the definition, basis of Legendre scaling function. This basis will be the basic of this paper in the following section. Legendre scaling function is used to approximate both state variables and control variables. Legendre scaling function can be defined as follows.

$$\Phi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{K/2} P_m(2^K t - 2n + 1), & \text{for } \dfrac{2n-2}{2^K} \le t \le \dfrac{2n}{2^K} \\ 0, & \text{Otherwise} \end{cases} \quad (5)$$

where P_m is the Legendre polynomial of order m; n refers to the section of time interval,
$n = 1, 2, \dots, 2^{K-1}$; K is the scaling parameter and can assume any positive integer and
$t \in [0,1]$. From this the Legendre polynomial can be given

$$P_m(t) = \frac{1}{2^m m!} \frac{d^m}{dt^m} (t^2 - 1)^m \quad (6)$$
$$P_0(t) = 1$$
$$P_1(t) = t$$
$$P_2(t) = \frac{1}{2}(3t^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5t^3 - 3t)$$
$$\vdots$$

4.1. Operational Matrix of Integration (OMI) for Legendre Scaling Function

Let P be an operational matrix of integration which is obtained from the integral of Legendre scaling function from 0 to t and these matrices play an important role to modeling the problems. According to [7] this is used to change the system of state equations into a set of algebraic equations which can be solved using software. This is represented as follows:

$$\int_0^t \Phi(\tau)d\tau = P\Phi(t) \qquad (7)$$

where $\Phi(t) =$

$$\left[\Phi_{10}(t)\ \ \Phi_{11}(t) \dots \Phi_{1M}(t)\ \ \Phi_{20}(t) \dots \Phi_{2M}\ \ \Phi_{2^{K-1}0} \dots \Phi_{2^{K-1}M}(t)\right]^T.$$

and P is called the cofficent matrix of $\Phi(t)$

$$P = \frac{1}{2^K}\begin{bmatrix} D & U \\ O & D \end{bmatrix}$$

where P is a $(2^{K-1}(M+1) \times 2^{K-1}(M+1))$ operational matrix of integration and O, D, U are $(M+1) \times (M+1)$ matrices and given by

$$U = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$D =$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{35}} & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \begin{array}{c}\frac{-1}{\sqrt{4M^2-16M+15}} \\ 0\end{array} & 0 & \frac{1}{\sqrt{4M^2-8M+3}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{4M^2-8M+3}} & 0 \end{bmatrix}$$

O is matrix that all entire elements are zeros.

Note: The integration of multiplication of Legendre scaling function and its transpose in the interval $t \in [0,1]$ is equal to identity matrix since Legendre scaling functions are orthonormal is as follows.

$$\int_0^1 \Phi(t)\Phi^T(t)\,dt = I_N \qquad (8)$$

where I_N, is identity matrix of dimension $N, (N = 2^{K-1}(M+1))$.

4.2. Continuity Test of Legendre Scaling Function

Scaling functions are not supported on whole interval $a \le x < b$; so these functions divide the interval of interest to number of sections depending on the value of scaling parameter K; for this reasons we have to add additional constraints to ensure the continuity of the state variables between different sections. There are $2^{K-1} - 1$ points at which the continuity of state variables.

$$t_i = \frac{i}{2^{K-1}} \quad , i = 1, 2, \dots, 2^{K-1} - 1$$

In addition, there is dimension of equality constraints is $(2^{K-1} - 1)s$ can be given as follows

$$(I_s \otimes \Phi')a = o_{(2^{K-1}-1)_{s\times 1}} \qquad (9)$$

where Φ' is continuity matrix

However, the dimension of matrix of continuity ensured constraints is $(2^{K-1} - 1) \times (2^{K-1}(M+1))$.

5.LINEAR TIME VARYING QUADRATIC OPTIMAL CONTROL PROBLEM

Before approximating the states variable and control variable, it is necessary to transform the time in the optimal control problem $t \in [t_0, t_f]$ into the interval $\tau \in [0,1]$; because Legendre scaling function [8] and Chebyshev scaling function [4] are defined on the interval $\tau \in [0,1]$.

Then the transformed optimal control problem is

$$\min J = t_f \int_0^1 (x^T(t)Qx(t) + u^T(t)Ru(t))d\tau \qquad (10)$$

$$\text{subject to} \quad \frac{dx(t)}{d\tau} = t_f(Ax(t) + Bu(t))$$

5.1. Approximation of State and Control Variable Using Legendre and Chebyshev Scaling Function

The approximation of state variables and control variables for the problem of linear time varying quadratic optimal control problems with linear constraint using Legendre and Chebyshev scaling function is as follows

$$x_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} a^i{}_{nm}\Phi_{nm}(t) \ , \quad \text{where} \quad i = 1, 2, 3, \dots, s$$

$$u_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} b^i{}_{nm}\Phi_{nm}(t) \ , \quad \text{where} \quad i = 1, 2, 3, \dots, r$$

These equations can be expanding to

$$\begin{aligned} x(t) = {} & a_{10}\ \Phi_{10}(t) + a_{11}\ \Phi_{11}(t) \\ & + \dots + a_{1M}\ \Phi_{1M}(t) \dots \\ & + \dots\ a_{2^{K-1}0}\Phi_{2^{K-1}0}(t) \\ & + \dots + a_{2^{K-1}M}\Phi_{2^{K-1}M}(t) \end{aligned}$$

$$\begin{aligned} u(t) = {} & b_{10}\ \Phi_{10}(t) + b_{11}\ \Phi_{11}(t) \\ & + \dots + b_{1M}\ \Phi_{1M}(t) \dots \\ & + \dots\ b_{2^{K-1}0}\Phi_{2^{K-1}0}(t) \\ & + \dots + b_{2^{K-1}M}\Phi_{2^{K-1}M}(t) \end{aligned}$$

These equations can be written in compact form are

$$\left.\begin{aligned} x(t) &= (I_s \otimes \Phi^T(t))a \\ u(t) &= (I_r \otimes \Phi^T(t))b \end{aligned}\right\} \qquad (11)$$

where I_s and I_r are sxs and rxr identity matrices respectively and $\Phi(t)$is $N \times 1$,

$$N = 2^{K-1}(M+1)$$

Vector of Legendre scaling function and unknown coefficients are given by:

$$\Phi(t) = [\Phi_{10}(t)\ \ \Phi_{11}(t) \dots \Phi_{1M}(t)\ \ \Phi_{20}(t) \dots \Phi_{2M} \dots \Phi_{2^{K-1}0}(t) \dots \Phi_{2^{K-1}M}(t)]^T$$

$$a = [a_{10}\ a_{11}\ a_{12}\ \dots\ a_{1M}\, a_{20}\ \dots a_{2M} \dots\ a_{2^{K-1}0}\ \dots\ a_{2^{K-1}M}]^T$$

and

$$b = [b_{10}\ b_{11}\ b_{12}\ \dots\ b_{1M}\, b_{20}\ \dots b_{2M} \dots\ b_{2^{K-1}0}\ \dots\ b_{2^{K-1}M}]^T$$

where a and b are vectors of unknown coefficients of dimension $sN \times 1$ and $rN \times 1$.

5.2. Approximation of Performance Index via Legendre and Chebyshev Scaling Function

We can approximate the performance index of linear time quadratic optimal control problems using Legendre and Chebyshev scaling function, substitute equations (11) into the objective function. We get

$$J = \frac{1}{2}\int_0^1 a^T(\Phi(t)\otimes I_s)Q(I_s\otimes\Phi^T(t))a + b^T(\Phi(t)\otimes I_r)R(I_r\otimes\Phi^T(t))b\,dt \quad (12)$$

By applying equation (8), we get the equation (12) and we can simplify this equation. We get

$$J = \tfrac{1}{2}(a^T(I_N\otimes Q)a + b^T(I_N\otimes R)b) \quad (13)$$

Moreover, equation (13) can be written in quadratic form is as follows

$$J = [a^T\quad b^T]\begin{bmatrix} I_N\otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & I_N\otimes R\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} \quad (14)$$

We can construct the quadratic form of linear time invariance optimal problems from the integration of state equation can be rewritten using Legendre and Chebyshev scaling function with operational matrix of integration, and unknown coefficients to be easily solved using MATLAB in the following form.

Now we can express the state equations in terms of the unknown parameters of the state variables and the control variables, state equation can be integrated as

From the integration of state equation can be rewritten using operational matrix of integration and unknown coefficients in the following form.

$$[(A\otimes P^T) - I_{Ns}\quad (B\otimes P^T)]\begin{bmatrix} a \\ b\end{bmatrix} = -g_0\delta \quad (15)$$

where $\delta = \frac{\sqrt{2}}{2^{K/2}}$, A is an $n\times n$ matrix and B is an $n\times m$ are the coefficient matrices of state variable and control variable from the state equation respectively, g_0 is the initial column vector, P is operational matrix of integration with the dimension of $2^{K-1}(M+1)$, I_{Ns} is identity matrix with the dimension of $2^{K-1}(M+1)$ and a, b are unknown coefficients.

By combining equations (9) and (15), we get the following form of equality constraints

$$\begin{bmatrix} (A\otimes P^T) - I_{Ns} & (B\otimes P^T) \\ I_s\otimes\Phi' & o_{(2^{K-1}-1)s\times Nr}\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} -g_0\delta \\ o_{(2^{K-1}-1)s\times 1}\end{bmatrix} \quad (16)$$

Then compact quadratic form is as follows:

$$\left.\begin{array}{ll} \min_z & z^T H z \\ \text{Subject to} & Fz = h \end{array}\right\} \quad (17)$$

where, $z = \begin{bmatrix} a \\ b\end{bmatrix}$

$$H = \begin{bmatrix} RR\otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & RR\otimes R\end{bmatrix}$$

$$F = \begin{bmatrix} (A\otimes P^T) - I_{Ns} & (B\otimes P^T) \\ (I_s\otimes\Phi') & o_{(2^{K-1}-1)s\times Nr}\end{bmatrix}$$

$$h = \begin{bmatrix} -g_0\delta \\ o_{(2^{K-1}-1)_{s\times 1}}\end{bmatrix}$$

The general steps to approximate state, control variable and performance indices using Legendre and Chebyshev scaling function with unknown coefficients of a and b.

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows.

$$a = [a_{10}\ a_{11} \dots a_{1M}\ a_{20}\ a_{21} \dots a_{2M} \dots a_{2^{K-1}0} \dots a_{2^{K-1}M}\ ]^T$$
$$b = [b_{10}\ \ b_{11} \dots b_{1M}\ \ b_{20}\ b_{21} \dots\ b_{2M}\ \dots\ b_{2^{K-1}0} \dots b_{2^{K-1}M}\ ]^T$$

Step (2): Generate the scaling function depending on K and M.

$$\Phi(t) = [\Phi_{10}(t)\ \ \Phi_{11}(t) \dots \Phi_{1M}(t)\ \Phi_{20}(t)\ \Phi_{21}(t) \dots \Phi_{2M}(t) \dots \Phi_{2^{K-1}0}(t) \dots\ \ \Phi_{2^{K-1}M}(t)]^T$$

Step (3): Approximate the state and control variables.

$$x(t) = \sum_{n=1}^{2^{K-1}}\sum_{m=0}^{M} a_{nm}\Phi_{nm}$$
$$u(t) = \sum_{n=1}^{2^{K-1}}\sum_{m=0}^{M} b_{nm}\Phi_{nm}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{2}}{2^{K/2}}\ \text{ and }\ \delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{(K-1)/2}},$$
$$\xi_0 = [x_i(0)\ 0\ 0 \dots 0\ |x_i(0)\ 0\ 0 \dots 0]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$\Phi' = [\Phi_{10}(t_i) \dots \Phi_{1M}(t_i)\ \Phi_{20}(t_i)\ \dots \Phi_{2M}(t_i) \dots\ \Phi_{2^{K-1}0}(t_i) \dots\ \ \Phi_{2^{K-1}M}(t_i)]$$

where $t_i = \frac{i}{2^{K-1}}$, i=1, 2, 3, . . ., $2^{K-1} - 1$

Step (6): Determine the quadratic programming problems.

Step (7): Solve the quadratic programming problem or equation (17) using MATLAB.

Example

Find the optimal control $u^*(t)$ and state variable $x^*(t)$ which minimize the performance index.

$$\min\quad J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt$$

Subject to $x'(t) = -x(t) + u(t),\quad x(0) = 1$

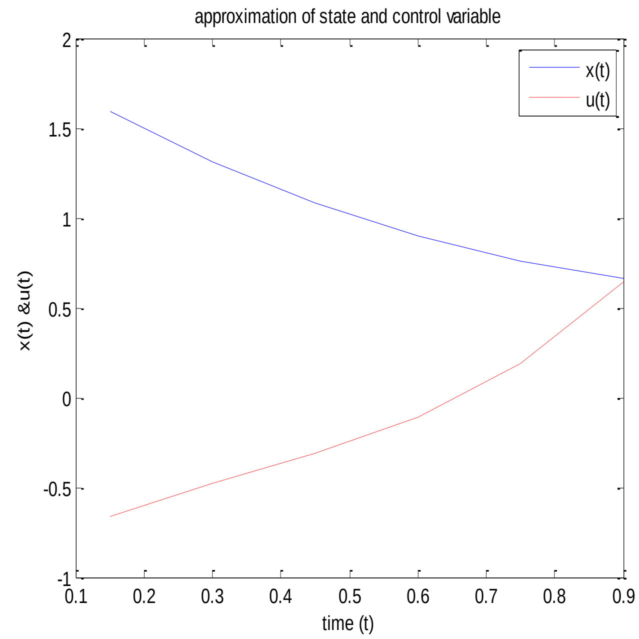

Figure 1. The approximation of state and control variable using Legendre scaling function

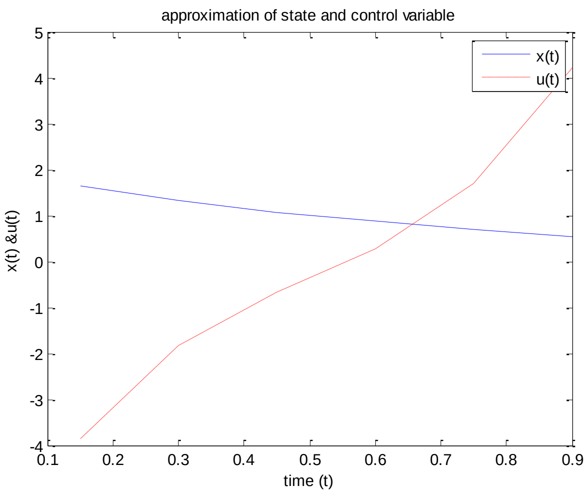

Figure 2. The approximation of state and control variable using Chebyshev scaling function

The approximation value of state variable and control variable in the linear time invariant system, were presented in Figure 1 and Figure 2 using Legendre scaling function and Chebyshev scaling function respectively. We have considered parameter values K=2, with polynomial of degree three. We observed that as the time interval, $t \in (0,1)$ the value of state variable decreases and control variable increases. MATLAB is used to plot the graphs to show the best result of problems. The graphs represent the best approximation at each system and the best value of state and control variable by the run considering numerical results of the solutions are shown in Figure 1and Figure 2.

Table 1. A comparison between Legendre and Chebyshev scaling function with the approximations of performance index of example

	Optimal value [9]		0.1929092981	
Order of poly	Performance index (J)			
	Legendre	Error	Chebyshev	Error
M=1	0.194179279497372	1.27×10^{-3}	0.196333244960920	3.42×10^{-3}
M=2	0.192998674286855	8.94×10^{-5}	0.193000989090520	9.17×10^{-5}
M=3	0.192909322011399	2.39×10^{-8}	0.192934130143056	2.48×10^{-5}

This example solved by using Legendre scaling function and Chebyshev scaling function. As M=1 to M=3 the approximation value is approaching to the optimal value obtained by [9]. From this table we conclude that Legendre scaling function is better than Chebyshev scaling function.

CONCLUSION

In this paper, we proposed numerical methods to solve linear time invariance quadratic optimal control problem by using Legendre and Chebyshev scaling function methods. Applying these methods, the linear time quadratic optimal control problem was converted into quadratic programming problem with in unknown coefficients and known scaling function and it has solved the quadratic programming problem using MATLAB. Furthermore, the proposed methods could be easily implemented in a MATLAB. The results obtained for the linear time optimal control systems with quadratic performance index using Legendre and Chebyshev scaling functions works for finding the approximation values of the state variable x(t), the control variable u(t) and performance indices. It was also deduced that the polynomial of scaling function with degree three and the time interval $t \in [0,1]$, the absolute error is less (almost zero) with the Legendre scaling functions than with the Chebyshev scaling functions method. Hence, the Legendre scaling function method is more suitable for solving the linear time invariance quadratic optimal control problem.

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