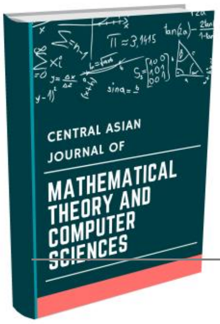

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A CRITICAL APPRAISAL OF INTUITIONIST FOUNDATIONS OF MATHEMATICS

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Abstract

The paper is a study of the intuitionist foundations of mathematic. It is work in the epistemology of mathematics. Hence, it is warranted by the need to ascertain the epistemic status of mathematical statements. Are they synthetic or a priori or synthetic a priori statements, as the case may be? The objective of the study is to evaluate the responses of intuitionist to these questions, so as to ascertain whether they could set to rest the controversies in the philosophy of mathematics. To achieve this objective, the method of content analysis has been adopted. This method is consistent with a qualitative research design. The major text selected for analysis are works of intuitionist mathematicians and other philosophers of mathematics. It has been submitted in the study, that although intuitionism is known for its proposal of the epistemology of subjective construction of mathematical entities, its system is inconsistent because of its dependence on ontological pluralism that leads to crude empiricism and the epistemological crisis of object absolutism.

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Introduction

The problem of traditional epistemology is the problem of object absolutism and that problem is equally evident in the intuitionist foundations of mathematical propositions. The affectation of intuitionism in this regard unfolds in a step-by-step analysis of the assumptions of the programme. Foundationally, intuitionism arose as a result of the crisis in the foundations of *Analysis*, regarding the antinomies of set theory. Some mathematicians had thought that the antinomies were functions of distorted comprehensions of mathematical entities. Hence, they resorted to seeking to found mathematics on a solid foundation defined and completely determined by a few manageable concepts that would not permit the incursion of any form of contradiction. To achieve this feat, Frege and his followers sought to understand mathematics in terms of logic, the principle of which are incapable of contradictions. Hilbert sought to articulate mathematics in terms of formal systems. The logical demonstration in the *Principia Mathematica* (1978) of Whitehead and Russell is a pure formalism. Russell had thought along the lines of Zermelo that: “The safe way to eliminate paradoxes of this type is to abandon the schema of comprehension” (Jech 2). The abandonment of this schema, which states that, if φ is a property, then there exists a set Y= {x: φ(x)} (Jech 2), is an abandonment of intuition. The lost of intuition or comprehension is the lost of meaning. But Russell saw no achievement in the abandonment of meaning. Hence, he reintroduced meaning through a policy called *The Doctrine of Types* (Russell, Principle 497). One thing was fundamental to these systems. They all sought to support mathematics without allowing any part of it to be lost.

On the contrary, intuitionism championed by the Dutch Mathematician E. J. Brouwer, accused the crisis in mathematics on improper pursuit of the discipline (Korner 119). So he sought to “build a new mathematics at all levels, by what he regarded as the truly mathematical methods” (Korner 119). To attain this, intuitionism first identifies the root of the improper pursuit of classical mathematics with the conflation of mathematics with mathematical language (Brouwer 2). The new mathematics would involve a total control of thought. The curtailing leads to a loss of a large part of mathematics, namely, the logico-linguistic aspect.

Brouwer traces the foundation of improper pursuit of mathematics to modifications in philosophical ideas about mathematics (1). The modifications led also to modifications in the “mechanisms of mathematical thought” (Brouwer 1). Such shift in paradigm could be identifiable with the developments in the history of philosophy. Ancient philosophical view of mathematics was coloured by Greek mysticism, according to which mathematical entities were esoteric or mystical objects. This idea is evident in the teachings of Pythagoras and Plato. Mathematical esoterism is known conventionally as Platonism. It is this form of thinking that Brouwer calls the independent existence of the object. The exaltation of the object is actually what traditional epistemology has been found difficult to abandon. Brouwer identifies it technically as the “observational standpoint” (1).

The philosophy of mathematics according to Brouwer maintains the observational standpoint with gradual modifications in mathematical operations. The philosophy that supported the operations of Eclidean geometry, the Guassian analysis and classical mathematics has been the observational standpoint. It baffles the intuitionist (Brouwer) that even after Kant had identified the importance of the subject in the project of mathematical pursuit, the observational standpoint persists.

The illegitimacy of this position in geometry became clear at the turn of the twentieth century when:

> ***… at the hand of a series of discoveries with which the names of Lobatchefsky, Bolyai, Reimann, Cayley, Klein, Hilbert, Einstein, Levi-Cavita and Hahn are associated, mathematics was gradually transformed into a mere science of numbers; and when beside observational space a great number of other spaces, sometimes exclusively originating from logical speculations, with properties distinct from the traditional but no less beautiful had found their arithmetic realization (Brouwer 3).***

The implication of this for classical geometry is that: “the science of classical (Euclidean, three-dimensional) space had to continue its existence as a chapter without priority …” (Brouwer 1).

The new wave of mathematics had implication for the members of the old formalist school. These scholars saw Euclidean geometry more as an applied science than as pure geometry. Thus, they sought to drive out of mathematics any element external to language and logic. The philosophical implication of this for mathematics was the blurring of the supposed distinction between mathematics and logic. The logico-linguistic method used exclusively in the new practice made mathematician to dream that if mathematics is reduced to pure logical operations on language, there is a possibility of the self-proof of its consistency by the system (Brouwer 2).

It is a pity that this philosophical view and its mechanism of mathematical thought have collapsed at the hand of discoveries made by Gödel in his second incompleteness theorem. Brouwer observes that what collapses is not only the linguistic mechanism but the notion of independent view of the object of study awaiting some observation to effect proof. According to him, legitimate mathematics is constructive. Constructivism is constructions of the object by the subject. Thus, the starting point and the end of mathematics is the activity of the subject. He faults any argument allowing for the existence of mathematics in the absence of the subject.

The activity of legitimate mathematics is a mental operation on the a priori intuition of time. This mathematics according to Brouwer is in need of no proof. It is a self-sufficient mathematical system (Korner 19). Its statements are synthetic arising from introspection. Unfortunately, Brouwer's proposal is capable of leading to solipsism.

According to Brouwer, a school he identifies as pre-intuitionism came so close to this position but could not overcome the influence of the observational standpoint in their study of the continuum. He writes as thus "On … occasions they seem to have introduced the continuum by having recourse to some logical axiom of existence, such as the 'axiom of ordinal connectedness,' or the 'axiom of completeness' without either sensory or epistemological evidence" (Brouwer, 2). He accused them of the application of classical logical principles without reserve, especially the principle of the excluded middle.

The First Act of Intuitionism

Brouwer's intuitionism is aimed at controlling and directing mathematics to the actual experience of its objects. Thus, it seeks to present a mathematics that is not distorted by the use of language and classical logic. This exercise Brouwer assigns to the first act of intuitionism. What the above sentence means is that intuitionism has more than one act. It actually has two acts. The first act aims at the separation of analytic mathematical language from mathematical statements as synthetic. The second act helps to protect the idea of the continuum and its consequent mathematical operations.

Thus, the first act of intuitionism focuses on:

> ***Completely separating mathematics from mathematical language and hence from the phenomenon of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics (Brouwer, 4).***

Brouwer tend to create the impression that mathematics arises from the experience of events in their succession. But he immediately takes away the empirical foundation by his reference to the common substratum of all twoities, which form the basic intuition of mathematics (Brouwer 4). What Brouwer drives at is the need to found mathematics on the a priori intuition of time championed by Kantian epistemology. The a priori intuition of time for Brouwer, refers to the succession of the perception of individuals in their binary frame of empirical representation as identity and difference. For instance, the perception of *this* and then *that*, or *those already perceived* and *this other*.

Intuitionism is of the view that by separating mathematics from mathematical language, the observational space and time would be obviated. To properly understand the pursuit of Brouwer and his followers it is important to know what his understanding of the project of classical mathematics is all about.

The intuitionists take the foundations of classical mathematics to be space and time (Brouwer 1). But their critique of this project is founded on the ontological status of the objects. Classical mathematics as Brouwer would want his readers to believe is plagued by what he calls the observational standpoint (Brouwer 1). The observational standpoint as indicated above refers to the independence of the object from the subject. Here, the subject also suffers lack of acquaintance with all the objects. Thus, it used the law of excluded middle and other principles of classical logic to seek to proof their existence, especially when it concerns the infinite entities. Dealing basically with logical principles on meaning, this system, Brouwer claims plunges itself into the paradoxes of language and logic. The problem is founded on the assumption of independently existing objects and relations of the Platonist sort.

To overcome this problem, Brouwer and the intuitionists seek to demonstrate what the real status of the object is. Non-Eclidean geometry has shown that space is not uniquely Eclidean. Thus "… the science of classical (Euclidean, three-dimensional) space had to continue its existence as a chapter without priority, on the one hand of the aforesaid (exact) science of numbers, on the other hand (as applied mathematics) of (naturally approximative) descriptive natural science (Brouwer 2). The idea of Newtonian absolute time which allows for extra-mental ontological status is also rejected. But the results of the new physics is never used by the intuitionists. The reason is yet unknown. The intuitionists retains the idea of the absolute in the consideration of time. They however divest time of all external qualities and make the substratum of all twoities the foundation of mathematics. This substratum according to them is Kantian. Thus, Brouwer argues that mathematics is founded on the concepts of unity, difference and counting. What is counted is the stored moments of distinct perception of discrete individuals, the status of which is intra-mental.

The experience is an introverted experience (Brouwer 7). The inner experiences is roughly sketched as thus:

twoity;

twoity stored and preserved aseptically by memory;

twoity giving rise to the conception of invariable unity;

twoity and unity giving rise to the conception of unity plus unity;

threeity as twoity plus unity, and the sequence of natural numbers;

Mathematical system conceived in such a way that a unity is a mathematical system and that two mathematical systems, stored and aseptically preserved by memory, apart from each other, can be added; etc (Brouwer 7).

The transition achieved in the above analysis is one of moving from the independent object of the observational standpoint to the active subject of intuitionist constructivism. Brouwer has showed his displeasure over undermining the subject's contribution to mathematics in classical mathematics (Brouwer 1). He laments that the introduction of the subjective side in Kantian epistemology had no serious effect on the modification of the mechanism of mathematical thought.

First Act of Intuitionism as Finitism

The intuitionist foundations of mathematics turn mathematical statement into mere report to ensure security. Reports about what? Reports on mental constructions of mathematical entities. The security for mathematical statements in the intuitionist system is founded on the ground that they are synthetic statements. They state the obvious, that is, they are reports on the state of mental constructions. To justify the statement is to look into consciousness and see. The search is carried out is in the mental life of the mathematician. The process of verification of mathematical statements is introspective. Thus intuitionist mathematics divides into three, namely: mathematical construction in the a priori intuition of time, introspective experience and reports on introspective experience. So, mathematics simply says what is there. There is, as such, no sense of any statement, except the sense in which it is the function of introspective experience.

Here again is logical atomism and crude empiricism smuggled in. Bertrand Russell describes the doctrine as "… a thoroughgoing empiricism …." (Principles x). He adds that if it is taken seriously, its consequences would "… even be more destructive than those recognized by its advocates" (Principle x). "Associated with this theory is the doctrine called finitism, which calls to question propositions involving infinite collections or infinite series, on the ground that such propositions are unverifiable" (Russell, Principles x). Thus, the problem faced by logical atomism, logical positivism, especially those who talk about protocol sentence in traditional epistemology is not different for intuitionism. This sentence repeats the sentence that started this paper. The world pictured by mathematical statements is the mental world. Brouwer knew that if his system were pursued consistently then all statements about the continuum would be nonsensical. After all, he had argued that there is no sense of the real number outside what is constructed. Thus, a real number is not identified with a real number generator, but in classical mathematics a real number could be define as a set of all real number generators that coincide with it. For instance, the classical Cauchy sequence is a real number generator: "… for every natural number k we can find a natural n = $n(k)$, such that $|a_{n+p} - a_n| < 1/k$ for every natural number p" (Heyting 16). The sequence shown is definitely $\{a_n\}$. Intuitionism decries such analysis. The k^{th} term referred to here is unknown. Thus the real number in question is unknown. It is on this basis that Heyting argues that the intuitionist rational number sequence is a real number generator. Consequently, he adds to the above sequence the following statement: "This must be so understood, that, given k, we are able to determine effectively $n(k)$" (16). The function "$n(k)$" means that the nth term varies directly as k. But even in the absence of any value, classical mathematics is capable of defining the sequence $\{b_n\}$ to be identical with $\{a_n\}$. "Two number-generator a = $\{a_n\}$ and b = $\{b_n\}$ are identical, if $a_n = b_n$ for every n" (Heyting 16). At this level the concept of coincidence is intuitionistically acceptable. But classically the coincidence is not the number generators but the coincidence of real numbers. The implication of this for classical mathematics is that the provision of $\{a_n\}$ and $\{b_n\}$ for any n is equivalent to the realization of the sequence. On the contrary, intuitionism would seek the presentation of the value. Thus, it is copy theoretic, allowing for the legitimacy of logical atomism.

What intuitionist synthetic or finitist mathematics seeks to achieve is the control of language to avoid paradoxes. Hence, it is not surprising that the idea of set theory is defined in terms of species and spread. Defined as a spread or species, the set simply represent a rule according to which elements are generated (Korner 128). Thus, the notion of the cardinality of the set becomes dependent on the actual generation of the elements. It is noteworthy that intuitionist logic or set theory does not transcend the realm of mathematics. Brouwer thought that it was the use of these concepts beyond this domain that affected its use in mathematics and led to the antinomies. So, the concepts of spread and species pertain only to numbers as generated naturally. The idea of choice sequence, which is an arbitrary determination of the first object of a sequence is the foundation of the spread concept. But once that has been done the spread rule is formed. "A spread law is a rule A which divides finite sequence of natural numbers into admissible and inadmissible sequences" (Korner 128). It is noteworthy that every sequence in classical mathematics is actually infinite. Hilbert would denote it as ideally infinite. But intuitionism keeps it at potential infinite. In intuitionism, however, the actual spread is finite. That explains the nature of the spread law. The continuum defines the natural numbers series. So, a second rule is used to complement the spread law. That law is called the complementary law. According to the intuitionists: "The complementary law m of a spread M assigns a definite mathematical entity to any finite sequence which is admissible according to the spread law M" (Korner 128). The facts of the spread are actual objects of constructions, to ensure mathematical finitism.

Furthermore, the species is defined as "… a property which mathematical entities can be supposed to possess. After a species S has been defined, any mathematical entity which has been or might have been defined before S was defined and satisfies the condition S, is a member of the species S" (Korner 130). The intuitionist idea of a species is such that only the element, which are not, themselves definitions of the species but satisfy such condition by virtue of their independent definitions are its elements. The intuitionist sets are rules or laws defined on elements already constructed. They are not principles used for construction. In this way, intuitionist believed they have avoided the problem of impredicativity that besiege naïve set theory.

The Second Act of Intuitionism

The finitist or verificationist orientation of intuitionism was also a matter of concern to the intuitionist. The implication of this is that the legitimacy of the intuitionists finitist talk makes pure nonsense of *Analysis* (that branch of mathematics that studies the continuum). Given such empiricism, the idea of the infinite becomes unacceptable (Russell, Principle x). As shown above any idea of sequence assumes intuitionist legitimacy once such sequence is finite. For instance, the notion of sequence referred to in the analysis of the twoity of time above is devoid of infinity. The reason is that no human being can live forever to reach the infinite. But the study of the continuum is not nonsensical. To overcome the consequence of the finitist implications of the first act of intuitionism, Brouwer proposes the second act of intuitionism. The second act of intuitionism defines the sequence on the basis of mathematical induction as follows:

> ***Admitting two ways of creating new mathematical entities: first in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired (so that, for decimal fractions having neither exact values, not any guarantee of ever getting exact values admitted); secondly in the shape of mathematics species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities, which have been defined to be 'equal to it (definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity) (Brouwer 6).***

The first refers to the notion of spread, whereas the second refers to that of species. But it is difficult to imagine the workability of the notion of induction in a strictly empiricist system. If the concept of logical idealization is denied then induction is not possible. What could be the substitute for induction is the notion of potential infinity. To accept the notion of potential infinity is to establish a policy. The intuitionist does just that. It is his belief that all classical statements about actual infinities established by logical principles are either promises of construction or research incentives. They do not refer to mathematical reality, which is essentially constructible. Intuitionism achieves this by undermining the status of logic.

The Intuitionist Logic

The idea of logic in intuitionist mathematics is that of a post factum record of principles used in mathematical construction (Korner 131). The intuitionist logic is, therefore, a mathematical logic, in the sense of being only the logic employed in the constructive activity of mathematics. The search for logical principles implies the investigation of mathematical construction (Korner 131). The status of necessity is taken away from intuitionist logic. The logic in question is purely contingent and its statements are reports on the operative structure of construction. In this wise, intuitionism opens the way for the expansion of logic as construction proceeds. Logic as a discipline for intuitionism, is a study of the structure of how humans have actually reasoned about specific facts. Hence, it is an empirical not a normative or an a priori discipline.

The report oriented view of logical and mathematical statements leads to the obviating from intuitionist logic of all the principles of logic that would aid existence statements to assume the validity of the observational standpoint. Such principles include the principles of the Excluded Middle, its attendant Bivalence and Double Negation.

The idea of logical proof in intuitionist logic is actually constructive. Logical statements are reports on these constructions. The operative constants in the intuitionist logic are divided into two categories, namely; the ones, the operations of which depend on no other operations and those, the operations of which depend on others. The constants of the first type "$\wedge, \vee$ and $\exists$" (Dummett 12). Those of the second type are "$\forall$ $\rightarrow$, and $\neg$" (Dummett 12). "A proof of $A \wedge B$ is anything that is a proof of *A and B*. A proof of $A \vee B$ is anything that is a proof of either *A or B*. A proof of $\exists x\ A(x)$ is anything that is a proof for some *n*, of the statements *A(n)*" (Dummett 12). Dummett puts it more elaborately as follows: "… a proof of $\exists x\ A(x)$ is a proof of some statement of the form *A(t)*, together with a proof that the object denoted by the term *t* belongs to the domain …." (24). "A proof of $\forall x\ A(x)$ is a construction of which we can recognize that, when applied to any number *n*, it yields a proof of *A(n)*" (Dummett 12). "A proof of $A \rightarrow B$ is a construction of which we can recognize that, applied to any proof of *A*, its yields a proof of *B*" (Dummett 13). "A proof of $\neg A$ is usually characterized as a construction of which we can recognize that, applied to any proof of *A*, it will yield a proof of a contradiction" (Dummett 13).

Another version of the analysis has been provided by Heyting. In what would follow very soon *P* and *Q* denote any proposition, *Q* is any given set, the variable x ranges over the objects of *S* and *P(x)* is a predicate of *x*.

Table 1: Showing intuitionistic logical forms

FORM OF ASSERTION	GROUNDS FOR THE ASSERTION
a) P or Q	a) At least one P, Q has already been proven
b) P and Q	b) Both P and Q have been Proven
c) P implies Q	c) One has a construction C of which it has been proved that whenever C is applied to any possible proof P then the result is a proof Q.
d) Not –P	d) The same as for "P implies 1#1", that is there is every possible proof of P is shown to be transformed into a contradiction.
e) There exists an x such that P(x)	e) One has constructed an s in S and proved P(s)
f) For all x P(x)	f) One has a proof which is shown to specialize to a proof P(s) for each s in S.

One fundamental thing about the two analysis is the different ways of exposing the uniqueness of intuitionist logic.

Critical Appraisal of Intuitionist Foundations of Mathematics

In intuitionism both logical and mathematical statements are synthetic. These statements are said to picture the mental world. The argument that traditional epistemology is copy theoretic becomes manifest in the above analysis. Thus, intuitionism like every other epistemic system before it is guilty of the absolute objective standpoints. The absolute objective standpoint is an epistemological fallacy that assumes that the object of experience is exclusively the foundations of legitimate knowledge claims. The assumption berates the contributions of the cognitive subject to knowledge claims in foundational analysis.

The above critique of intuitionism appears to be at variance with the intuitionist promotion of subjective experience in mathematical construction. But that is exactly the complication created by intuitionists, which has resulted in epistemic inconsistency in the system. Intuitionism countenances ontological pluralism. This pluralism is manifested in the acceptance of the multiplicity of individuals, which individuals in their unity (distinctness) and diversity are given in sense perception. A unit of discrete perception, which is the perception of a unique individual, is stored in consciousness, such that any other perception of an individual with un-identical qualifications is understood as a difference in perception, and it is equally stored and added to the previous memory of the first individual, to form a twoity. Another unique perception is added to the previous two to form a threeity and so on, until a continuum is formed. This is crude empiricism that limits mathematics to the perceptual plural ontology of individuals. Considering that this individuals that must first exist before perception are cognitive objects upon which mathematics depends, intuitionism could legitimately be charged for object absolutism. Hence, despite its promotion of a return to a philosophy of mathematics of subjective construction, intuitionism introduces inconsistency in its system by inadvertently importing object absolutism into it.

The domain of intuitionist foundations of mathematics is unintelligible. The Kantian idea of a priori intuition of time founded on Newtonian physics goes with that physics. The foundation of modern physics is the theory of relativity. Within this frame, experiment is demonstrated by virtue of the synchronization of time. The time of this physics is the relative time. Thus, just as the general theory of relativity justified the arguments of non-Euclidean geometries, the special theory of relativity modifies the concept of absolute time. Consequently any grand talk about absolute time is unintelligible. Such was the foundation of time countenanced by Kant. Intuitionist claim to be followers of Kant but criticize Newton's notion of time. Such an intellectual attitude breeds confusion.

Heyting sought to overcome this implication facing intuitionism by arguing that the time of Brouwer's analysis are instances of perception. Perception as shown by cognitive studies is continuous and flowing. The distinction of the elements of perception is a matter of attention. So, it is difficult to understand how perception makes possible the natural numbers, except multiple discontinuous perceptions, which gives attention to the same time as specified by clocks. What their suggestions imply is the fallacy of ontological convenience; a situation in epistemology, where one proposes some putative ontological domain as a solution to the skepticism resulting from object absolutism.

Conclusion

Intuitionism, like other schools in the foundations of mathematics is plagued by the search for the object as an exclusive indicator of legitimate knowledge claims. Despite Brouwer's attempt to incorporate the subject into the foundations of mathematics, his inadvertent commitment to the object, which is the bane of western epistemology, led to incoherence between his proposal and his philosophical analysis. Besides, the finitism proposed by intuitionist mathematics and logic has limited hitherto a priori disciplines to empirical sciences. Intuitionism is crude solipsistic empiricism and cannot therefore carry the weight of the foundations of mathematics. Be that as it may, its promotion of the inclusion of subjective contributions to foundational analysis is quite instructive.

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