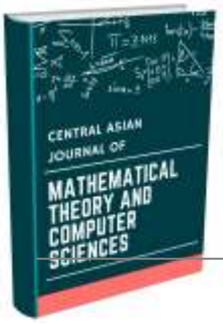

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The Effects of Weight-Doubling Sequences on the Compactness of Differences of Composition Operators on Bergman Spaces

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Abstract

Differences that are bounded and compact between two composition operators that are acting from the weighted Bergman space $A^{1+2\epsilon}_{\omega_m}$ to the Lebesgue space $L^{1+\epsilon}_{v_m}$, where $0 \leq \epsilon < \infty$ and ω_m class $\mathcal{D}$ of radial weights with a two-sided doubling requirement. New description of $(1+\epsilon)$-Carleson measures for $A^{1+2\epsilon}_{\omega_m}$, with $\epsilon > 0$ and $\omega_m \in \mathcal{D}$, Using discs of pseudohyperbolic geometry is proven. This last theorem generalizes the standard definition of $(1+\epsilon)$- Carleson uses weights as a baseline for the Bergman space he creates. $A^{1+2\epsilon}_{\alpha}$ with $0 < \epsilon < \infty$ to the setting of doubling weights. The case $\omega_m \in \widehat{\mathcal{D}}$ is also briefly talked about, and a question is raised about this case.

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1. Introduction and Main Results

Recently, the operators have become very popular due to its special applications in recent times in a wide range of sciences and engineering [1,2,3,4]. Especially, the composition operator [5,6,7,8,9,10,11,12,13] for a summary of the many studies that have been conducted on these operators in various function spaces.

In 2004, Nieminen and Saksman [14] showed that the compactness of $C_{\varphi_m} - C_{\psi_m}$ on the Hardy space.

In 2020 Choe, Choi, Koo and Yang [15] showed characterized compact operators by using Carleson measures Further, Moorhouse [16] gives characterized the compactness on the standard weighted Bergman space $A^2_{\epsilon-1}$. And others [17,18,19]

So, let's start by defining some terms For $0 \leq \epsilon < \infty$ and a positive Borel measure v_m on $\mathbb{D}$, the Lebesgue space $L^{1+\epsilon}_{v_m}$ consists of complex valued v_m-measurable functions f_m on $\mathbb{D}$ such that

$$\|f_m\|^{1+\epsilon}_{L^{1+\epsilon}_{v_m}} = \int_{\mathbb{D}} \sum |f_m(z)|^{1+\epsilon} dv_m(z) < \infty.$$

If v_m is continuous in $L^{1+\epsilon}_{v_m}$. See [20,21].

For a radial weight ω_m, write $\widehat{\omega}_m(z) = \int_{|z|}^{1} \sum \omega_m(s)ds$ for all $z \in \mathbb{D}$. In this paper we always assume $\widehat{\omega}_m(z) > 0$, for otherwise $A^{1+\epsilon}_{\omega_m} = \mathcal{H}(\mathbb{D})$ for each $0 \leq \epsilon < \infty$.if there exist $K = K(\omega_m) > 1$ and $C = C(\omega_m) > 1$ such that $\widehat{\omega}_m(1+\epsilon) \geqslant C\widehat{\omega}_m\left(1 - \frac{1-(1+\epsilon)}{K}\right)$ for all $-1 \leqslant \epsilon < 0$, then we write $\omega_m \in \widetilde{\mathcal{D}}$. In other words, $\omega_m \in \widetilde{\mathcal{D}}$ if there exists $K = K(\omega_m) > 1$ and $C' = C'(\omega_m) > 0$ such that

$$\widehat{\omega}_m(1+\epsilon) \leqslant C' \int_{1+\epsilon}^{1-\frac{1-(1+\epsilon)}{K}} \sum \omega_m(1+\epsilon)d(1+\epsilon), \ -1 \leqslant \epsilon < 0.$$

The intersection $\widehat{\mathcal{D}} \cap \widetilde{\mathcal{D}}$ is denoted by $\mathcal{D}$, see [22].

In this work from the Bergman weighting scheme, we think about compact differences between two composition operators. space $A^{1+\epsilon}_{\omega_m}$ to the Lebesgue space $L^{1+2\epsilon}_{v_m}$ when $0 \leq \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$. To state the first main result, write

$$\delta_m(z) = \sum \frac{\psi_m(z) - \varphi_m(z)}{1 - \overline{\psi_m(z)}\varphi_m(z)}, \ z \in \mathbb{D}.$$

The next result generalizes [[18], Theorem 1.2] to the setting of doubling weights.

Theorem 1. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let v_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. Then the following statements are equivalent:

(i) $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is bounded;

(ii) $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is compact;

(iii) $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are compact (or equivalently bounded) from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{v_m}$.

The proof of Theorem 1 We first show that $C_{\varphi_m} - C_{\psi_m}$ is compact if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded. The proof of this implication is straightforward and relies on the fact that the norm of $f_m \in \mathcal{H}(\mathbb{D})$ in $A^{1+2\epsilon}_{\omega_m}$ is comparable to the $L^{1+2\epsilon}_{\omega_m}$-norm of the non-tangential maximal function $(f_m)(z) = \sup_{\zeta \in \Gamma(z)} |f_m(\zeta)|$, where

$$\Gamma(z) = \left\{\zeta \in \mathbb{D}: |\theta - \arg \zeta| < \frac{1}{2}\left(1 - \frac{|\zeta|}{1+\epsilon}\right)\right\}, \ z = (1+\epsilon)e^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\}$$

We first observe that for each ρ-lattice $\{z_k\}$ the function

$$z \mapsto \sum_k \sum (a+\epsilon)^m_k \left(\frac{1-|z_k|}{1-\overline{z_k}z}\right)^M \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{1}{1+2\epsilon}}}$$

belongs to $A^{1+2\epsilon}_{\omega_m}$ for all $(a+\epsilon)_m = \{(a+\epsilon)^m_k\} \in \ell^{1+2\epsilon}$ and its $A^{1+2\epsilon}_{\omega_m}$-norm is dominated by a universal constant times $\|(a+\epsilon)_m\|_{\ell^{1+2\epsilon}}$.Then, Khinchine's inequality is used in conjunction with this testing function. A complete characterization of such measures in the case $\omega_m \in \widehat{\mathcal{D}}$ can be found in [23], [24], [25], [26]. In particular, it is known that if $\epsilon > 0$ and $\omega_m \in \widehat{\mathcal{D}}$, then μ_m is a $(1+\epsilon) -$ Carleson measure for $A^{1+2\epsilon}_{\omega_m}$ if and only if the function

$$\zeta \mapsto \int_{\Gamma(\zeta)} \sum \frac{d\mu_m(z)}{\omega_m(T(z))} \quad (1.1)$$

belongs to $L_{\omega_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$. Here and from now on $T(z) = \{\zeta \in \mathbb{D}: z \in \Gamma(\zeta)\}$ is the tent induced by $z \in \mathbb{D} \setminus \{0\}$. Further, $\omega_m(E) = \int_E \sum \omega_m dA$ for each measurable set $E \subset \mathbb{D}$.

It is well known that $\Delta(a, 1+\epsilon)$ is an Euclidean disk centered at $(1-(1+\epsilon)^2)a/(1-(1+\epsilon)^2|a|^2)$ and of radius $(1-|a|^2)(1+\epsilon)/(1-(1+\epsilon)^2|a|^2)$. We denote $\tilde{\omega}_m(z) = \hat{\omega}_m(z)/(1-|z|)$ for all $z \in \mathbb{D}$ and note that

$$\|f_m\|_{A_{\tilde{\omega}}^{1+2\epsilon}}^{1+} = \|f_m\|_{A_m^{1+2\epsilon}}, \; f_m \in \mathcal{H}(\mathbb{D}), \quad (1.2)$$

provided $\omega_m \in \mathcal{D}$, by [22]. Our embedding theorem generalizes the case $n = 0$ of [[10], Theorem 1] to doubling weights and reads as follows.

Theorem 2. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let μ_m be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) μ_m is a $(1+\epsilon)$-Carleson measure for $A_{\omega_m}^{1+2\epsilon}$;

(ii) $I: A_{\omega_m}^{1+2\epsilon} \to L_{\mu_m}^{1+\epsilon}$ is compact;

(iii) the function

$$\Theta_{\mu_m}^{\omega}(z) = \sum \frac{\mu_m(\Delta(z, 1+\epsilon))}{\omega_m(S(z))}, \; z \in \mathbb{D} \setminus \{0\}$$

belongs to $L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ for some (equivalently for all) $-1 < \epsilon < 0$.

Moreover,

$$\| \; I \; \|_{A_m^{1+\epsilon} \to L_{\mu_m}^{1+\epsilon}}^{1+\epsilon} = \|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}_m}}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \quad (1.3)$$

We may not replace $L_{\tilde{\omega}_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ by $L_{\omega_m}^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ in part (iii) of Theorem 2. A counter example can beconstructed as follows. Write $D(z, 1+\epsilon)$ for the Euclidean disc$\{\zeta: |\zeta - z| < 1+\epsilon\}$. Let $(1+\epsilon)_n = 1 - 2^{-n}$ and $A_n = D(0, (1+\epsilon)_{n+1}) \setminus D(0, (1+\epsilon)_n)$ for all $n \in \mathbb{N}$. Pick up an $\omega_m \in \mathcal{D}$ such that it vanishes on A_{2n} for all $n \in \mathbb{N}$. A simple example of a such weight is $\sum_{n \in \mathbb{N}} \chi_{A_{2n+1}}$. Then choose μ_m such that (35)

for some $\varepsilon > 0$ its support is contained in the union of the discs $\Delta(a_n, \varepsilon)$ which have the property that for some fixed $-1 < \epsilon < 0$ we have $\Delta(z, 1+\epsilon) \subset A_{2n}$ for all $z \in \Delta(a_n, \varepsilon)$ and for all $n \in \mathbb{N}$. The choice $a_n = ((1+\epsilon)_{2n} + (1+\epsilon)_{2n+1})/2$ works if $-1 < \epsilon < 0$ and $\varepsilon = \varepsilon(1+\epsilon) > 0$ are sufficiently small. Then, for such an $(1+ \; \epsilon \;)$, the norm $\|\Theta_{\mu_m}^{\omega_m}\|_{L_{\omega_m}^{(1+2\epsilon)-(1+\epsilon)}}$ vanishes and thus it cannot be comparable to $\|\Theta_{\mu_m}^{\omega_m}\|_{\frac{L}{(1+2\epsilon\epsilon-(1+\epsilon)}}$ which is non-zero if μ_m is not a zero measure because $\tilde{\omega}_m$ is strictly positive. Moreover, by choosing μ appropriately the norm $\|\Theta_{\mu_m}^{\omega_m}\|_{L_{\tilde{\omega}m}\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$ can be made infinite.

It partially completes [13]'s primary result for class D alone. [27] demonstrated an analogous result for Hardy spaces. Theorem [11] closes the minor Bergman space gap between Hardy and conventional weighted Bergman spaces.

Theorem 3. Let $0 \leq \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$, and let v_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. Then $C_{\varphi_m} - C_{\psi_m}: A^{1+\epsilon}_{\omega_m} \to L^{1+2\epsilon}_{v_m}$ is bounded (resp. compact) if and only if δC_{φ_m} and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+2\epsilon}_{v_m}$.

If $\epsilon = 0$ then the boundedness (resp. compactness) of $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ implies the same property for $C_{\varphi_m} - C_{\psi_m}$ by Proposition 4 below. Further, Proposition 5 below shows that $C_{\varphi_m} - C_{\psi_m}$ is compact if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded when $\epsilon > 0$. But we do not know if the boundedness of $C_{\varphi_m} - C_{\psi_m}$ implies that of $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ if $\omega_m \in \widehat{\mathcal{D}} \setminus \mathcal{D}$ if $\epsilon \geqslant 0$.

2. Carleson Measures

If $\omega_m \in \mathcal{D}$, then there exist constants $0 < \alpha = \alpha(\omega_m) \leqslant \beta = \beta(\omega_m) < \infty$ and $C = C(\omega_m) \geqslant 1$ such that

$$\frac{1}{C}\left(\frac{1-(1+\epsilon)}{1-(1+2\epsilon)}\right)^{\alpha} \leqslant \frac{\widehat{\omega}_m(1+\epsilon)}{\widehat{\omega}_m(1+2\epsilon)} \leqslant C\left(\frac{1-(1+\epsilon)}{1-(1+\epsilon)}\right)^{\beta}, \ -1 \leqslant \epsilon < 0 \qquad (2.1)$$

the class $\mathcal{D}$ because the right hand inequality is satisfied if and only if $\omega_m \in \widehat{\mathcal{D}}$ by [24], Lemma 2.1], while the left hand inequality describes the class $\tilde{\mathcal{D}}$ in an analogous manner [22], (2.27)].

Proof of Theorem 2. If μ_m is a $(1+\epsilon)$-Carleson measure for $A^{1+2\epsilon}_{\omega_m}$, then $I: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{\mu_m}$ is automatically compact by [26], Theorem 3(iii)]. Therefore, it suffices to show that (i) and (iii) are equivalent and establish (1.3). $|f_m|^{1+\epsilon}$, Fubini's theorem,Hölder's inequality and (1.2) imply $\|f_m\|^{1+\epsilon}_{L^{1+\epsilon}_m}$

$$\lesssim \int_{\mathbb{D}} \sum\left(\int_{\Delta(z,1+\epsilon)} \frac{|f_m(\zeta)|^{1+\epsilon}}{(1-|\zeta|^2)^2} dA(\zeta)\right) d\mu_m(z)$$

$$= \int_{\mathbb{D}} \sum\left(\int_{\Delta(z,1+\epsilon)} |f_m(\zeta)|^{1+\epsilon} \frac{\tilde{\omega}_m(\zeta)}{\hat{\omega}_m(\zeta)(1-|\zeta|)} dA(\zeta)\right) d\mu_m(z)$$

$$\leqslant \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{\mu_m(\Delta(\zeta, 1+\epsilon))}{\omega_m(S(\zeta))} \tilde{\omega}_m(\zeta) dA(\zeta) \leqslant \sum \|f_m\|^{1+\epsilon}_{A^{1+2\epsilon}_{\omega_m}} \|\Theta^{\omega_m}_{\mu_m}\|_{L^{(1+2\epsilon)(1+\epsilon)}_{\tilde{\omega}_m}}$$

$$= \sum \|f_m\|^{1+2\epsilon}_{A^{1+22\epsilon}_{\omega}} \|\Theta^{\omega_m}_{\mu_m}\|_{L^{(1+2\epsilon)(1+\epsilon)}_{\tilde{\omega}_m}}, \ f_m \in A^{1+22\epsilon}_{\omega_m}$$

Therefore μ_m is a $(1+\epsilon)$-Carleson measure for $A^{1+2\epsilon}_{\omega_m}$ and $\| I \|^{1+\epsilon}_{A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{\mu_m}} \lesssim$

$\|\Theta^{\omega_m}_{\mu_m}\|_{L_{\tilde{\omega}}^{\frac{1+2\epsilon}{}}}$.

Conversely, assume that μ_m is a $(1+\epsilon)$-Carleson measure for $A^{1+2\epsilon}_{\omega_m}$. Then (1.2) shows that μ_m is also a $(1+\epsilon)$-Carleson measure for $A^{1+2\epsilon}_{\tilde{\omega}_m}$ and the corresponding operator norms are comparable. Further, since $\omega_m \in \mathcal{D}$ by the hypothesis, an application of (2.1) shows that $\tilde{\omega}_m \in \mathcal{D}$. Therefore [23], Theorem 1(a)] implies

$$\|B^{\tilde{\omega}_m}_{\mu_m}\|^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}_{L^{(1+2\epsilon)-(1+\epsilon)}_{\tilde{\omega}_m}} = \int_{\mathbb{D}} \sum \left(\int_{\Gamma(z)} \frac{d\mu_m(\zeta)}{\tilde{\omega}_m(T(\zeta))}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \tilde{\omega}_m(z) dA(z) < \infty,$$

where

$$\Gamma(z) = \left\{\zeta \in \mathbb{D}: |\arg\zeta - \arg z| < \frac{1}{2}\left(1 - \frac{|\zeta|}{|z|}\right)\right\}, \ z \in \mathbb{D} \setminus \{0\}$$

is a non-tangential approach region with vertex at z. Further, by [26], Theorem 3(iii)] we have $\| I \|^{1+\epsilon}_{A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{\mu_m}} \gtrsim \|B^{\tilde{\omega}_m}_{\mu_m}\|_{L^{(1+2\epsilon)-(1+\epsilon)}_{\tilde{\omega}_m}}$.

Let now $-1<\epsilon<0$ be given. For $\epsilon>0$ and $z\in\mathbb{D}\setminus D\left(0,1-\frac{1}{K}\right)$ write $z_K=(1-(1+\epsilon)(1-|z|))e^{i\arg z}$. Pick up $(1+\epsilon)=(1+\epsilon)(1+\epsilon)>1$ and $(1+2\epsilon)=(1+2\epsilon)(1+\epsilon)\in\left(1-\frac{1}{K},1\right)$ sufficiently large such that $\Delta(z_K,1+\epsilon)\subset\Gamma(z)$ for all $z\in\mathbb{D}\setminus D(0,R)$. Straightforward applications of the left hand inequality in (2.1) show that $\tilde{\omega}_m(T(\zeta))\lesssim\omega_m(S(\zeta))$, as $|\zeta|\to 1^-$, and $\tilde{\omega}_m(z)\lesssim\tilde{\omega}_m(z_K)$ for all $z\in\mathbb{D}\setminus D(0,R)$. In an analogous way we deduce $\omega_m(S(\zeta))\lesssim\omega_m(S(z_K))$ for all $\zeta\in\Delta(z_K,1+\epsilon)$ and $z\in\mathbb{D}\setminus D(0,(1+2\epsilon))$ by using the right hand inequality.

$$\infty\quad>\| I \|^{1+\epsilon}_{A^{1+2\epsilon}_{\omega m}\to L^{1+\epsilon}_m}\gtrsim\int_{\mathbb{D}\setminus D(0,R)}\left(\int_{\Delta(z_K,1+\epsilon)}\frac{d\mu_m(\zeta)}{\omega_m(T(\zeta))}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}\tilde{\omega}_m(z)dA(z)$$

$$\geq\int_{\mathbb{D}\setminus D(0,R)}\left(\frac{\mu_m(\Delta(z_K,1+\epsilon))}{\omega_m(S(z_K))}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}\tilde{\omega}_m(z_K)dA(z)$$

$$=\int_{\mathbb{D}\setminus D(0,1-K(1-R))}\left|\Theta^{\omega_m}_{\mu_m}(z)\right|^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}\tilde{\omega}_m(z)dA(z)$$

3. Sufficient Conditions

In this section we establish sufficient conditions for $C_{\varphi_m}-C_{\psi_m}\colon A^{1+\epsilon}_{\omega_m}\to L^{1+2\epsilon}_{v_m}$ to be bounded or compact. All these results are valid under the hypothesis $\omega_m\in\widehat{\mathcal{D}}$ despite the main results stated in the introduction concern only the class $\mathcal{D}$. We begin with the case $\epsilon\geq 0$.

Proposition 4. Let $0<\epsilon<\infty$ and $\omega_m\in\widehat{\mathcal{D}}$, and let v_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+2\epsilon}_{v_m}$, then $C_{\varphi_m}-C_{\psi_m}\colon A^{1+\epsilon}_{\omega_m}\to L^{1+2\epsilon}_{v_m}$ is bounded (resp. compact).

Proof. We begin with the statement on the boundedness. Let first $\epsilon>0$. Let $f_m\in A^{1+\epsilon}_{\omega_m}$ with $\|f_m\|^{1+1}_{A^1_{\omega}+\epsilon}\leqslant 1$ Fix $-1<\epsilon<0$, and denote $E=\{z\in\mathbb{D}\colon|\delta_m(z)|<1+\epsilon\}$ and $E'=\mathbb{D}\setminus E$. Write

$$(C_{\varphi_m}-C_{\psi_m})(f_m)=(C_{\varphi_m}-C_{\psi_m})(f_m)\chi_{E'}+(C_{\varphi_m}-C_{\psi_m})(f_m)\chi_E$$

and observe that it is enough to prove that the quantities are bounded.

$$\left\|(C_{\varphi_m}-C_{\psi_m})(f_m)\chi_{E'}\right\|_{L^{1+2\epsilon}_{v_m}}\text{ and }\left\|(C_{\varphi_m}-C_{\psi_m})(f_m)\chi_E\right\|_{L^{1+2\epsilon}_{v_m}}\qquad(3.1)$$

We begin with considering the first quantity in (3.1). By the definition of the set E we have the estimate

$$\sum\left|(C_{\varphi_m}-C_{\psi_m})(f_m)\chi_{E'}\right|\leqslant\frac{1}{1+\epsilon}\sum\left(\left|\delta_m C_{\varphi_m}(f_m)\right|+\left|\delta_m C_{\psi_m}(f_m)\right|\right)\qquad(3.2)$$

on $\mathbb{D}$. Since the operators $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ both are bounded from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+2\epsilon}_{v_m}$ by the hypothesis, the first term in (3.1) is bounded by (3.2).

We next show that also the second term in (3.1) is bounded. Let μ_m be a finite nonnegative Borel measure on $\mathbb{D}$ and h_m a measureable function on $\mathbb{D}$. For an analytic self-map φ_m of $\mathbb{D}$, the weighted pushforward measure is defined by

$$(\varphi_m)_*(h_m,\mu_m)(M)=\int_{\varphi_m^{-1}(M)}\sum h_m d\mu_m\qquad(3.3)$$

for each measurable set $M\subset\mathbb{D}$. If μ_m is the Lebesgue measure, we omit the measure in the notation and write $(\varphi_m)_*(h_m)(M)$ for the left hand side of (3.3). By the measure theoretic change of variable [28]. Section [29], we have $\|\delta_m C_{\varphi_m}(f_m)\|_{L^{1+2\epsilon}_{v^1_m}}=\|f_m\|_{L^{1+2\epsilon}_{(\varphi_m)_*(|\delta_m|1+2\epsilon v_m)}}$ for each $f_m\in A^{1+\epsilon}_{\omega_m}$. Therefore the identity operator from $A^{1+2\epsilon}_{\omega^{1+2\epsilon}_m}$ to $L^{1+2\epsilon}_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon}v_m)}$ is bounded by the hypothesis. Hence $(\varphi_m)_*(|\delta_m|^{1+2\epsilon}v_m)(\Delta(\zeta,(1+2\epsilon)))\lesssim\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}$ for all $\zeta\in\mathbb{D}\setminus\{0\}$ by [23], (35) Volume (3) December 2022; [UBJSR: ISSN [1858-6139]: (Online)

Theorem 1(c)]. Further, by 10 , Lemma 3], with $\omega_m \equiv 1$ and $\epsilon = 0$, there exists a constant $C = C((1+\epsilon), (1+\epsilon), (1+2\epsilon)) > 0$ such that

$$|f_m(z) - f_m(a)|^{1+2\epsilon} \leqslant C \frac{\rho(z,a)^{1+\epsilon}}{(1-|a|)^2} \int_{\Delta(a,1+2\epsilon)} \sum |f_m(\zeta)|^{1+\epsilon} dA(\zeta),\ a \in \mathbb{D},\ z \in \Delta(a, 1+\epsilon)$$

for all $f_m \in A^{1+\epsilon}_{\omega_m}$ with $\|f_m\|^{1+\epsilon}_{A^{1+\epsilon}_{\omega}} \leqslant 1$. This and Fubini's theorem yield

$$\begin{aligned}
&\| (C_{\varphi_m} - C_{\psi_m})(f_m)\chi_E \|_{L^{1+2\epsilon}_{v_m^{1+\epsilon}}} \\
&= \int_E \sum \left|f_m(\varphi_m(z)) - f_m(\psi_m(z))\right|^{1+2\epsilon} v_m(z) dA(z) \\
&\leq \int_E \sum \frac{|\delta_m(z)|^{1+2\epsilon}}{(1-|\varphi_m(z)|)^2} \int_{(\varphi_m(z),1+2\epsilon)} |f_m(\zeta)|^{1+2\epsilon} dA(\zeta) v_m(z) dA(z) \\
&\leqslant \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \left(\int_{\varphi_m^{-1}(\Delta(\zeta,1+2\epsilon))\cap E} \frac{|\delta_m(z)|^{1+2\epsilon}}{(1-|\varphi_m(z)|)^2} v_m(z) dA(z) \right) dA(\zeta) \\
&\leq \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \left(\int_{\varphi_m^{-1}(\Delta(\zeta,1+2\epsilon))} \frac{|\delta(z)|^{1+2\epsilon}}{(1-|\zeta|)^2} v_m(z) dA(z) \right) dA(\zeta) \\
&= \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta) \\
&\lesssim \int_{\mathbb{D}} \sum |f_m(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta) = \int_{\mathbb{D}} \sum_{1+2\epsilon} |f_m(\zeta)|^{1+2\epsilon} d\mu_m(\zeta).
\end{aligned} \tag{3.4}$$

Standard arguments show that $\mu_m(S(a)) \lesssim \omega_m(S(a))^{\frac{1+2\epsilon}{1+\epsilon}}$ for all $a \in \mathbb{D} \setminus \{0\}$. Hence [23], Theorem 1(c)] yields $\sum \|(C_{\varphi_m} - C_{\psi_m})(f_m)\chi_E\|^{1+2\epsilon}_{L^{1+2\epsilon}_{v_m}} \lesssim \sum \|f_m\|^{1+2\epsilon}_{L^{1+2\epsilon}_{\mu_m^1}} \lesssim \| \sum f_m \|^{1+2\epsilon}_{A^{1+\epsilon}_{\omega m}}$. Therefore also the second term in (3.1) is bounded. This finishes the proof of the case $\epsilon > 0$.

Let now $\epsilon = 0$. By following the proof above, it suffices to show that

$$\int_S \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta,1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta) \lesssim \sum \omega_m(S) \tag{3.5}$$

for every Carleson square $S \subset \mathbb{D}$. By the hypothesis, the identity operator from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)}$ is bounded, and hence $(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S) \lesssim \omega_m(S)$ for all S by [[23], Theorem 1(b)]. But for each positive Borel measure μ_m on $\mathbb{D}$, Fubini's theorem yields

$$\begin{aligned}
&\int_{S(a)} \sum \frac{\mu_m(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta) \\
&= \int_{\{z\in\mathbb{D}: S(a)\cap\Delta(z,1+2\epsilon)\neq\emptyset\}} \sum \left(\int_{S(a)\cap\Delta(z,1+2\epsilon)} \frac{dA(\zeta)}{(1-|\zeta|)^2} \right) d\mu_m(z) \\
&\leqslant \int_{S((a+\epsilon)_m)} \sum \left(\int_{\Delta(z,1+2\epsilon)} \frac{dA(\zeta)}{(1-|\zeta|)^2} \right) d\mu_m(z) = \mu_m(S((a+\epsilon)_m)),\ |a| > (1+2\epsilon) \quad ,
\end{aligned} \tag{3.6}$$

where $(a+\epsilon)_m = (a+\epsilon)_m(a, 1+2\epsilon) \in \mathbb{D}$ satisfies $\arg(a+\epsilon)_m = \arg a$ and $1 - |(a+\epsilon)_m| = 1 - |a|$ for all $a \in \mathbb{D} \setminus \overline{D(0,1+2\epsilon)}$. By applying this to $\mu_m = (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)$ and using the hypothesis $\omega_m \in \widehat{\mathcal{D}}$ we deduce (35) .

$$\int_{S(a)} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta) \lesssim \sum (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)\big(S((a+\epsilon)_m)\big)$$
$$\lesssim \sum \omega_m\big(S((a+\epsilon)_m)\big) \lesssim \sum \omega_m(S(a)),\ |a| > (1+2\epsilon).$$

This estimate implies (3.5), and thus the case $\epsilon = 0$ is proved.

To obtain the compactness statement, it suffices to show that the quantities

$$\|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L^{1+2\epsilon}_{v_m}},\ \text{and}\ \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|_{L^{1+2\epsilon}_{v_m}} \qquad (3.7)$$

tend to zero as $n \to \infty$ for each sequence $\{(f_m)_n\}_{n\in\mathbb{N}}$ in $A^{1+\epsilon}_{\omega_m}$ which tends to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and satisfies $\|(f_m)_n\|_{A^{1+\epsilon}_{\omega_m}} \leqslant 1$ for all $n \in \mathbb{N}$. Since $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are compact from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+2\epsilon}_{v_m}$ by the hypothesis, an application of (3.2) to $f_m = (f_m)_n$ shows that the first quantity in (3.7) tends to zero as $n \to \infty$. As for the second quantity, observe that (3.4) implies

$$\sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|^{1+2\epsilon}_{L^{1+2\epsilon}_{v_m}}$$
$$\lesssim \int_{\mathbb{D}} \sum |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)\big(\Delta(\zeta, 1+2\epsilon)\big)}{(1-|\zeta|)^2} dA(\zeta),\ n \in \mathbb{N}. \qquad (3.8)$$

by the hypothesis, we have $(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S(\zeta))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} \to 0$ as $|\zeta| \to 1^-$by [19. Theorem 3(ii)]. Now, for each $\zeta \in \mathbb{D} \setminus \{0\}$ pick up $\zeta' = \zeta'(\zeta, 1+2\epsilon) \in \mathbb{D}$ such that $\arg \zeta' = \arg \zeta, \Delta(\zeta', 1+2\epsilon) \subset S(\zeta)$ and $1 - |\zeta'| = 1 - |\zeta|$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. Then

$$\sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(S(\zeta))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \geqslant \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)\big(\Delta(\zeta', 1+2\epsilon)\big)}{\omega_m(S(\zeta))^{\frac{1+\epsilon}{1+2\epsilon}}}$$
$$\gtrsim \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)\big(\Delta(\zeta', 1+2\epsilon)\big)}{\omega_m\big(S(\zeta')\big)^{\frac{1+2\epsilon}{1+\epsilon}}},\ \zeta \in \mathbb{D} \setminus \{0\}$$

and hence $\sup_{\zeta\in\mathbb{D}\setminus\{0\}} (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} < \infty$ and, for a given $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) \in (0,1)$ such that $\sum (\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))/\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}} < \varepsilon$ for all $\zeta \in \mathbb{D} \setminus D(0,\eta)$. Further, by the uniform convergence, there exists $N = N(\varepsilon, \eta, 1+2\epsilon) \in \mathbb{N}$ such that $|(f_m)_n|^{1+2\epsilon} \leqslant \varepsilon$ on $D(0,\eta)$ for all $n \geqslant N$. These observations together with the proof of the boundedness case and (3.8) yield

$$\sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|^{1+2\epsilon}_{L^{1+2\epsilon}_{v_m}}$$
$$\lesssim \sup_{\zeta\in D(0,\eta)\setminus\{0\}} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \int_{D(0,\eta)} |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta)$$
$$+ \sup_{\zeta\in\mathbb{D}\setminus D(0,\eta)} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+2\epsilon} v_m)(\Delta(\zeta, 1+2\epsilon))}{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}} \int_{\mathbb{D}\setminus D(0,\eta)} |(f_m)_n(\zeta)|^{1+2\epsilon} \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta)$$
$$\lesssim \varepsilon \int_{\mathbb{D}} \sum \frac{\omega_m(S(\zeta))^{\frac{1+2\epsilon}{1+\epsilon}}}{(1-|\zeta|)^2} dA(\zeta) + \varepsilon \sum \|(f_m)_n\|^{1+2\epsilon}_{A^{1+\epsilon}_{\omega m}} \leqslant \varepsilon,\ n \geqslant N.$$

Thus also the second quantity in (3.7) tends to zero as $n \to \infty$ in the case $\epsilon > 0$.

Finally, let $\epsilon = 0$. The compactness of the identity operator from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)}$ implies $(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)(S(\zeta))/\omega_m(S(\zeta)) \to 0$ as $|\zeta| \to 1^-$by (35) .

[[26], Theorem 3(ii)]. By following the proof above the only different step consists of making the quantity

$$J(\eta) = \int_{\mathbb{D}\setminus D(0,\eta)} \sum |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta)$$

Standard arguments can now be used to make the right hand side smaller than a pregiven $\varepsilon > 0$ for η sufficiently large by using $(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)(S(\zeta))/\ \omega_m(S(\zeta)) \to 0$ as $|\zeta| \to 1^-$, see, for example, [25 pp.[9],[27] for details. This completes the proof of the proposition.

The next result is a counter part of Proposition 4 in the case $\epsilon > 0$.

Proposition 5. Let $0 \le \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$, and let ν_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $\delta_m C_{\varphi_m}$ and δC_{ψ_m} are bounded from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{\nu_m}$, then $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{\nu_m}$ is compact.

Proof. Let $\{(f_m)_n\}$ be a bounded sequence in $A^{1+2\epsilon}_{\omega_m}$ such that $(f_m)_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{\nu_m}$ by the hypothesis, they are also compact by [26], Theorem 3(iii)], and therefore

$$\lim_{n\to\infty} \sum \left(\|\delta_m (f_m)_n(\varphi_m)\|_{L^{1+\epsilon}_{\nu_m}} + \|\delta_m (f_m)_n(\psi_m)\|_{L^{1+\epsilon}_{\nu_m}} \right) = 0 \qquad (3.9)$$

Let $-1 < \epsilon < 1$, and denote $E = \{z \in \mathbb{D}: |\delta(z)| < 1+\epsilon\}$ and $E' = \mathbb{D} \setminus E$. To prove the compactness of $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{\nu_m}$, it suffices to show that

$$\lim_{n\to\infty} \sum \left(\|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|_{L^{1+\epsilon}_{\nu_m}} + \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L^{1+\epsilon}_{\nu_m}} \right) = 0$$

since

$$\begin{aligned} &\sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\|^{1+\epsilon}_{L^{1+\epsilon}_{\nu_m}} \\ &= \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E\|^{1+\epsilon}_{L^{1+\epsilon}_{\nu_m}} \\ &+ \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|^{1+\epsilon}_{L^{1+\epsilon}_{\nu_m}} \end{aligned}$$

By using (3.2) and (3.9), it is easy to show that

Further, by (3.4), we have

$$\lim_{n\to\infty} \sum \|(C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_{E'}\|_{L^{1+\epsilon}_{\nu_m}} = 0$$

$$\sum \| (C_{\varphi_m} - C_{\psi_m})((f_m)_n)\chi_E \|^{1+\epsilon}_{L^{1+\epsilon}_{\nu_m}} \lesssim \int_{\mathbb{D}} \sum |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)(\Delta(\zeta, 1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta).$$

Let $\varepsilon > 0$. Since the identity operator from $A^{1+2\epsilon}_{\omega_m}$ to $L_{(\varphi_m)_*(|\delta_m|^{1+\epsilon}\nu_m)}$ is bounded by the hypothesis,[[26]. Theorem 3(iii)] and the dominated convergence theorem imply the existence of an $(1+2\epsilon)_0 = (1+2\epsilon)_0(\varepsilon) \in (0,1)$ such that

$$\int_{\mathbb{D}} \sum \left(\int_{\Gamma(z)\setminus\overline{D(0,(1+2\epsilon)_0)}} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)(\Delta(\zeta,1+2\epsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right)^{\varepsilon^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}} \omega_m(z)dA(z)$$
$$< \varepsilon^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}. \qquad (3.10)$$

Further, by the uniform convergence, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|(f_m)_n(z)| < \varepsilon^{\frac{1}{1+\epsilon}}$ for all $n \geqslant N$ and $z \in \overline{D(0,(1+2\epsilon)_0)}$. Therefore, for all $n \geqslant N$, by Fubini's theorem, Hölder's inequality,[[25]. Lemma 4.4] and (3.10), we have

$$\sum \left\| \left(C_{\varphi_m} - C_{\psi_m}\right)((f_m)_n)\chi_E \right\|_{L^{1+\epsilon}_{v_m}}^{1+\epsilon}$$
$$\lesssim \sum \left(\int_{\overline{D(0,(1+2\epsilon)_0)}} + \int_{\mathbb{D}\setminus\overline{D(0,(1+2\epsilon)_0)}} \right) |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)(\Delta(\zeta,1+2\epsilon))}{(1-|\zeta|)^2} dA(\zeta)$$
$$\lesssim \varepsilon + \int_{\mathbb{D}} \sum \left(\int_{\Gamma(z)\setminus\overline{D(0,(1+2\epsilon)_0)}} |(f_m)_n(\zeta)|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)(\Delta(\zeta,1+2\epsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right) \omega_m(z)dA(z)$$
$$\leqslant \varepsilon + \int_{\mathbb{D}} \sum N((f_m)_n)^{1+\epsilon}(z) \left(\int_{\Gamma(z)\setminus\overline{D(0,(1+2\epsilon)_0)}} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)(\Delta(\zeta,1+2\epsilon))}{\omega_m(T(\zeta))(1-|\zeta|)^2} dA(\zeta) \right) \omega_m(z)dA(z)$$
$$\leqslant \varepsilon \sum \left(1 + \| N((f_m)_n) \|_{L^{1+\epsilon}_{\omega}}^{1+\epsilon} \right) = \varepsilon$$

Therefore $\lim_{n\to\infty} \sum \left\| \left(C_{\varphi_m} - C_{\psi_m}\right)((f_m)_n)\chi_E \right\|_{L^{1+\epsilon}_{v_m}} = 0$, and thus $C_{\varphi_m} - C_{\psi_m}$ is compact from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{v_m}$.

4. Necessary Conditions

In this section we establish necessary conditions for $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ to be bounded or compact.

Proposition 6. Let either $0 \le \epsilon < \infty$ and $\omega_m \in \widehat{\mathcal{D}}$ or $\epsilon = 0$ and $\omega_m \in \mathcal{D}$, and let v_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m analytic self-maps of $\mathbb{D}$. If $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is bounded (resp. compact), then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are bounded (resp. compact) from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{v_m}$.

Proof. Let first $\epsilon > 0$ and $\omega_m \in \widehat{\mathcal{D}}$. We begin with the boundedness and show in detail that $\delta_m C_{\varphi_m}$ is bounded from $A^{1+\epsilon}_{\omega_m}$ to $L^{1+2\epsilon}_{v_m}$. For each $a \in \mathbb{D}$, consider the function

$$(f_m)_a(z) = \left(\frac{1-|a|}{1-\bar{a}z}\right)^{\gamma} \sum \omega_m(S(a))^{-\frac{1}{1+\epsilon}}, \; z \in \mathbb{D}$$

induced by ω_m and $0 \le \epsilon < \infty$. Then[[24], Lemma 2.1] implies that for each $\gamma = \gamma(\omega_m, 1+2\epsilon) > 0$ sufficiently large we have $\|(f_m)_a\|_{A_{\omega_m^{1+2}}} = 1$ for all $a \in \mathbb{D}$. Fix such a γ. Since $C_{\varphi_m} - C_{\psi_m}$ is bounded, we have

$$1 = \sum \|(f_m)_a\|_{A^{1+2\epsilon}_{\omega_m}}^{1+\epsilon} \gtrsim \left\| \left(C_{\varphi_m} - C_{\psi_m}\right)((f_m)_a) \right\|_{L^{1+\epsilon}_{v_m^1}}^{1+\epsilon}$$
$$= \int_{\mathbb{D}} \sum \left| \left(\frac{1-|a|}{1-\bar{a}\varphi_m(z)}\right)^{\gamma} - \left(\frac{1-|a|}{1-\bar{a}\psi_m(z)}\right)^{\gamma} \right|^{1+\epsilon} \frac{v_m(z)}{\omega_m(S(a))^{\frac{1+\epsilon}{1+2\epsilon}}} dA(z)$$
$$= \int_{\mathbb{D}} \sum \left| \frac{1-|a|}{1-\bar{a}\varphi_m(z)} \right|^{\gamma(1+\epsilon)} \left| 1 - \left(\frac{1-\bar{a}\varphi_m(z)}{1-\bar{a}\psi_m(z)}\right)^{\gamma} \right|^{1+\epsilon} \frac{v_m(z)}{\omega_m(S(a))^{\frac{1+\epsilon}{1+2\epsilon}}} dA(z).$$

According to[[17], p. 795], for each $0 \le \epsilon < \infty$ and $-1 < \epsilon < 0$ there exist a constant $C = C(\gamma, 1+\epsilon) > 0$ such that

$$\sum \quad \left|1-\left(\frac{1-\bar{a}z}{1-\bar{a}w_m}\right)^{\gamma}\right| \geqslant \sum \ C|a|\rho(z,w_m),\ z\in\Delta(a,1+\epsilon),\ a,w_m\in\mathbb{D}. \qquad (4.1)$$

An application of this inequality gives (35) .

$$1 \gtrsim \int_{\varphi_m^{-1}(\Delta(a,1+\epsilon))} \sum \frac{|a|^{1+\epsilon}|\delta_m(z)|^{1+\epsilon}}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}} v_m(z)dA(z)$$
$$= |a|^{1+\epsilon} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon}v_m)(\Delta(a,1+\epsilon))}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}}.$$

It follows that $(\varphi_m)_*(|\delta_m|^{1+\epsilon}v_m)$ is a bounded $(1+\epsilon)$-Carleson measure for $A^{1+2\epsilon}_{\omega_m}$ by [[23], Theorem 1(c)], and hence $\delta_m C_{\varphi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is bounded. The same argument shows that also $\delta_m C_{\psi_m}$ is bounded.

For the compactness statement, first observe that $(f_m)_a$ tends to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Then, if $C_{\varphi_m} - C_{\psi_m}$ is compact, we have $\lim_{|a|\to 1^-} \sum \left\|\left(C_{\varphi_m} - C_{\psi_m}\right)((f_m)_a)\right\|_{L^{1+\epsilon}_{v_m}} = 0$. By arguing as above we deduce

$$\lim_{|a|\to 1^-} \sum \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon}v_m)(\Delta(a,1+\epsilon))}{(\omega_m(S(a)))^{\frac{1+\epsilon}{1+2\epsilon}}} = 0$$

Therefore $\delta_m C_{\varphi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is compact by[[26]. Theorem 3]. The same argument shows that also $\delta_m C_{\psi_m}$ is compact.

Let now $\epsilon = 0$ and $\omega_m \in \mathcal{D}$. The statement follows from the proof above with the modification that [[13]. Theorem 2], valid for $\omega_m \in \mathcal{D}$, is used instead of [[23]. Theorem 1(c)] and[[26]. Theorem 3]. The only extra step is to observe that for each $\omega_m \in \mathcal{D}$ there exists $1+\epsilon = (1+\epsilon)(\omega_m) \in (0,1)$ such that $\omega_m(S(a)) = \omega_m(\Delta(a,1+\epsilon))$ for all $a \in \mathbb{D}\setminus\{0\}$. This follows from (2.1). With this guidance we consider the proposition proved.

The next result establishes a counter part of Proposition 6 when $\epsilon > 0$.

Proposition 7. Let $0 \le \epsilon < \infty$ and $\omega_m \in \mathcal{D}$, and let v_m be a positive Borel measure on $\mathbb{D}$. Let φ_m and ψ_m be analytic self-maps of $\mathbb{D}$. If $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is bounded, then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\psi_m}$ are both bounded from $A^{1+2\epsilon}_{\omega_m}$ to $L^{1+\epsilon}_{v_m}$.

Proof. Let $\{z_k\}_{k\in\mathbb{N}}$ be a ρ-lattice such that it is ordered by increasing modulii and $z_k \neq 0$ for all k. Then by [[30], Theorem 1] there exist constants $M = M(1+2\epsilon,\omega_m) > 1$ and $C = C(1+2\epsilon,\omega_m) > 0$ such that the function

$$F_m(z) = \sum_k \sum \ (a+\epsilon)_k^m \left(\frac{1-|z_k|}{1-\overline{z_k}z}\right)^M \frac{1}{\left(\omega_m\left(T(z_k)\right)\right)^{\frac{1}{1+2\epsilon}}},\ z\in\mathbb{D},$$

belongs to $A^{1+2\epsilon}_{\omega_m}$ and satisfies $\sum\|F_m\|_{A^{1+2\epsilon}_{\omega_m}} \leqslant C\sum\|(a+\epsilon)_m\|_{\ell^{1+2\epsilon}}$ for all $(a+\epsilon)_m = \{(a+\epsilon)_k^m\} \in \ell^{1+2\epsilon}$. Since $C_{\varphi_m} - C_{\psi_m}: A^{1+2\epsilon}_{\omega_m} \to L^{1+\epsilon}_{v_m}$ is bounded by the hypothesis, we deduce

$$\|(a+\epsilon)_m\|_{\ell}^{1+2\epsilon} \gtrsim \|F_m\|^{1+\epsilon}_{A^{1+2\epsilon}_{\omega_m}} \gtrsim \int_{\mathbb{D}} \sum\left|\left(C_{\varphi_m} - C_{\psi_m}\right)\circ F_m(z)\right|^{1+\epsilon} dv_m(z)$$

$$= \int_{\mathbb{D}} \sum \left| \sum_k (a+\epsilon)_k^m \left(\left(\frac{1-|z_k|}{1-\overline{z_k}\varphi_m(z)}\right)^M - \left(\frac{1-|z_k|}{1-\overline{z_k}\psi_m(z)}\right)^M \right) \frac{1}{\left(\omega\left(T(z_k)\right)\right)^{\frac{1}{1+2\epsilon}}} \right|^{1+\epsilon} dv_m(z),\ (a+\epsilon)_m \in \ell^{1+2\epsilon}.$$

We now replace $(a+\epsilon)_k^m$ by $(a+\epsilon)_k^m(1+\epsilon)_k(1+\epsilon)$, integrate with respect to $-1 < \epsilon < 0$, and then apply Fubini's theorem and Khinchine's inequality to get (35)

$$\|(a+\epsilon)_m\|_{\ell^{1+2\epsilon}}^{1+\epsilon} \gtrsim \int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^2 \left| \left(\frac{1-|z_k|}{1-\overline{z_k}\varphi_m(z)}\right)^M - \left(\frac{1-|z_k|}{1-\overline{z_k}\psi_m(z)}\right)^M \right|^2 \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{2}{1+2\epsilon}}} \right)^{\frac{1+\epsilon}{2}} dv_m(z),\ (a+\epsilon)_m \in \ell^{1+2\epsilon}.$$

By applying (4.1) and the estimate $|1-\overline{z_k}z| = 1-|z_k|$, valid for all $z \in \Delta(z_k,\rho)$ and $k \in \mathbb{N}$, we obtain

$$\sum \left| \left(\frac{1-|z_k|}{1-\overline{z_k}\varphi_m(z)}\right)^M - \left(\frac{1-|z_k|}{1-\overline{z_k}\psi_m(z)}\right)^M \right| = \sum \left|\frac{1-|z_k|}{1-\overline{z_k}\varphi_m(z)}\right|^M \left|1-\left(\frac{1-\overline{z_k}\varphi_m(z)}{1-\overline{z_k}\psi_m(z)}\right)^M\right|$$

$$\gtrsim |z_k| \sum |\delta_m(z)| \left|\frac{1-|z_k|}{1-\overline{z_k}\varphi_m(z)}\right|^M \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z)$$

$$= |z_k| \sum |\delta_m(z)| \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z),\ z \in \mathbb{D},\ k \in \mathbb{N},$$

and hence

$$\|(a+\epsilon)_m\|_{\ell^{1+2\epsilon\epsilon}}^{1+\epsilon} \gtrsim |z_1|^{1+\epsilon} \int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z) \right)^{\frac{1+\epsilon}{2}} dv_m(z). \quad (4.2)$$

If $\epsilon \geqslant 0$ then the inequality $\sum_j c_j^x \leqslant \left(\sum_j c_j\right)^x$, valid for all $c_j \geqslant 0$ and $x \geqslant 1$, imply $\|(a+\epsilon)_m\|_{\ell}^{1+2\epsilon}$

$$\gtrsim \int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^{1+\epsilon} |\delta_m(z)|^{1+\epsilon} \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{1+\epsilon}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z) \right) dv_m(z). \quad (4.3)$$

To get the same estimate for $-2 < \epsilon < 0$ we apply Hölder's inequality. It together with the fact that the number of discs $\Delta(z_k, 1+\epsilon)$ to which each $\varphi_m(z)$ may belong to is uniformly bounded yields

$$\int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^{1+\epsilon} |\delta_m(z)|^{1+\epsilon} \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{1+\epsilon}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z) \right) dv_m(z)$$

$$\leqslant \int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k,1+\epsilon))}(z) \right)^{\frac{1+\epsilon}{2}}$$

$$\cdot \left(\sum_k \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z) \right)^{1-\frac{1+\epsilon}{2}} dv_m(z)$$

$$\lesssim \int_{\mathbb{D}} \sum \left(\sum_k |(a+\epsilon)_k^m|^2 |\delta_m(z)|^2 \frac{1}{\left(\omega_m(T(z_k))\right)^{\frac{2}{1+2\epsilon}}} \chi_{\varphi_m^{-1}(\Delta(z_k,\rho))}(z) \right)^{\frac{1+\epsilon}{2}} dv_m(z)$$

Thus (4.3) holds for each $0 \leq \epsilon < \infty$. By using Fubini's theorem we now deduce

$$\|(a+\epsilon)_m\|_{\ell^{1+2\epsilon}}^{1+\epsilon} \gtrsim \sum_k \sum |(a+\epsilon)_k^m|^{1+\epsilon} \frac{1}{\left(\omega_m\left(T(z_k)\right)\right)^{\frac{1+\epsilon}{1+2\epsilon}}} \int_{\varphi_m^{-1}(\Delta(z_k,\rho))} |\delta(z)|^{1+\epsilon} dv_m(z)$$

$$= \sum_k \sum |(a+\epsilon)_k^m|^{1+\epsilon} \frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)\left(\Delta(z_k,\rho)\right)}{\left(\omega_m\left(T(z_k)\right)\right)^{\frac{1+\epsilon}{1+\epsilon\epsilon}}}, \ (a+\epsilon)_m \in \ell^{1+2\epsilon}.$$

Therefore the sequence belongs to $\left(\ell^{\frac{1+2\epsilon}{1+\epsilon}}\right)^* \simeq \ell^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$, and consequently

$$\sum \left\{\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)\left(\Delta(z_k,\rho)\right)}{\left(\omega_m\left(T(z_k)\right)\right)^{\frac{1+\epsilon}{1+2\epsilon}}}\right\}_{k\in\mathbb{N}}$$

$$\sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v)\left(\Delta(z_k,\rho)\right)}{\left(\omega_m\left(T(z_k)\right)\right)^{\frac{1+\epsilon}{1+2\epsilon}}}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} < \infty.$$

Pick up an $1+\epsilon = (1+\epsilon)(\rho) \in (0,1)$ such that $\Delta(z,\rho) \subset \Delta(z_k, 1+\epsilon)$ for all $z \in \Delta(z_k,\rho)$ and $k \in \mathbb{N}$. The right hand inequality in (2.1) shows that $\widehat{\omega}_m(z) = \widehat{\omega}_m(z_k)$ and $\omega_m(S(z)) = \omega_m\left(S(z_k)\right)$ for all $z \in \Delta(z_k,\rho)$ and $k \in \mathbb{N}$. Then, as $\{z_k\}_{k\in\mathbb{N}}$ is a ρ-lattice, we deduce

$$\sum_k \int_{\Delta(z_k,\rho)} \sum \left(\frac{(\varphi_m)_*\left(|\delta_m|^{1+\epsilon v_m}\right)(\Delta(z,\rho))}{\omega_m(S(z))}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \widetilde{\omega}_m(z) dA(z)$$

$$= \sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)\left(\Delta(z_k, 1+\epsilon)\right)}{\omega_m\left(S(z_k)\right)}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} \widetilde{\omega}_m(z_k)(1-|z_k|)^2$$

$$\leqslant \sum_k \sum \left(\frac{(\varphi_m)_*(|\delta_m|^{1+\epsilon} \vartheta_m)\left(\Delta(z_k, 1+\epsilon)\right)}{\left(\omega_m\left(S(z_k)\right)\right)^{\frac{1+\epsilon}{1+2\epsilon}}}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}}$$

$$\lesssim \sum_k \sum \left(\frac{(\varphi_m)_*\left(|\delta_m|^{1+\epsilon v_m}\right)\left(\Delta(z_k,\rho)\right)}{\left(\omega_m(T(z_k))\right)^{\frac{1+\epsilon}{1+2\epsilon}}}\right)^{\frac{1+2\epsilon}{(1+2\epsilon)-(1+\epsilon)}} < \infty.$$

Therefore $(\varphi_m)_*(|\delta_m|^{1+\epsilon} v_m)$ is a $(1+\epsilon)$-Carleson measure for $A_{\omega_m}^{1+2\epsilon}$ by Theorem 2 For the same reason, $(\psi_m)_*(|\delta_m|^{1+\epsilon} v_m)$ is a $(1+\epsilon)$-Carleson measure for $A_{\omega_m}^{1+2\epsilon}$. The proof is complete.

5. Proofs of Main Theorems

The key outcomes presented in the introduction readily follow from the propositions demonstrated in the previous two parts.

Proof of Theorem 11 The theorem follows by Propositions 5 and 7 Namely, if $\delta_m C_{\varphi_m}$ and $\delta_m C_{\varphi_m}$ are bounded from $A_{\omega_m}^{1+2\epsilon}$ to $L_{v_m}^{1+\epsilon}$, then $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \to L_{v_m}^{1+\epsilon}$ is compact, and thus bounded as well, by Proposition 5. Conversely, if $C_{\varphi_m} - C_{\psi_m}: A_{\omega_m}^{1+2\epsilon} \to L_{v_m}^{1+\epsilon}$ is bounded, then $\delta_m C_{\varphi_m}$ and $\delta_m C_{\varphi_m}$ are bounded by Proposition 7 , Proof of Theorem 3 proof Propositions 4 and 6 imply the theorem.

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