MONOPHONIC DISTANCE

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Mathematics, Algebra Graph

Abstract

A u v path is monophonic if it has no chords for any two vertices u and v in a connected graph G, and the monophonic distance \( d_m(u, v) \) is the length of the longest u v monophonic path in G. The monophonic eccentricity of each vertex v in G is given by \( e_m(v) = \max d_m(u, v) : u V \). It is demonstrated that the monophonic center of a graph exists in every graph. The subgraph created by the vertices of G exhibiting minimal monophonic eccentricity is the monophonic center of G. Additionally, it is demonstrated that each connected graph G monophonic center is located within one of its blocks.

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1. INTRODUCTION

We take a finite connected graph with no loops and many edges, denoted by \( G = (V(G), E(G)) \). The letters p and q, respectively, stand for the order and size of G. The length of the shortest u v path in G is used to define the distance \( d(u, v) \) between u and v.

The radius \( rad G \) of G is the vertices of G is minimum, eccentricity, and the diameter \( diam G \) of G is the maximum eccentricity among the vertices of G.

1.1. Definition. An edge \( u_i u_j \) with \( j \geq i + 2 \) is a chord of a path \( u_1, u_2, \ldots, u_n \) in a connected graph G. If a u v path P is chord less, it is referred to as a monophonic path. The monophonic distance from u to v, abbreviated as \( dm(u, v) \), is the length of the longest u v monophonic path. A u v monophonic path is defined as one whose length is equal to \( dm(u, v) \).

1.2. Example. In the graph G given in Figure 1.1, we can easily check that \( d(v_1, v_4) = 2, D(v_1, v_4) = 6 \), and \( dm(v_1, v_4) = 4 \). The monophonic path \( P : v_1, v_2, v_3, v_4 \) is \( v_1 - v_4 \) monophonic while the monophonic path \( Q : v_1, v_3, v_4 \) is not \( v_1 - v_4 \) monophonic.

The usual distance \( d \) are metrics on the vertex set \( V \) of a connected graph G, whereas the monophonic distance \( dm \) not based on metrics \( V \). To get the graph G given in Fig 1.1, \( dm(v_4, v_6) = 5, dm(v_5, v_6) = 1 \) and \( dm(v_5, v_6) = dm(v_4, v_6) \). Hence \( dm(v_4, v_6) > dm(v_4, v_5) + dm(v_5, v_6) \) and so the triangle inequality is not satisfied.
1.3. Result. Let $u$ and $v$ be two vertices in a graph $G$. Then

1. $d_m(u, v) = 0$ if and only if $u = v$.
2. $d_m(u, v) = 1$ if and only if $uv$ is an edge of $G$.
3. $d_m(u, v) = p - 1$ if and only if $G$ is the path with endvertices $u$ and $v$.
4. $d_m(u, v) = d_m(u, v) = D(u, v)$ if and only if $G$ is a tree.

1.4. Definition. The monophonic eccentricity of each vertex $v$ in a connected graph $G$ is given by $e_m(v) = \max\{d_m(u, v) : u \in V\}$. A monophonic eccentric vertex of $v$ is one where $d_m(u, v) = e_m(v)$ for the vertex $u$ of $G$. The formulas $\text{rad}_m G = \min\{e_m(v) : v \in V\}$ and $\text{diam}_m G = \max\{e_m(v) : v \in V\}$, respectively, determine the monophonic radius and diameter of $G$.

1.5. Example. We will use a condensed explanation in this example, as indicated in Table 1.1. The graph $G$ provided is shown in along with the vertices' eccentricities and monophonic distances in a monophonic manner. Note that $\text{rad}_m G = 3$ and $\text{diam}_m G = 5$.

$$
\begin{array}{ccccccccc}
& v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & e_m(v) \\
v_1 & 0 & 1 & 1 & 4 & 1 & 4 & 3 & 4 & 4 \\
v_2 & 1 & 0 & 4 & 3 & 1 & 5 & 4 & 1 & 5 \\
v_3 & 1 & 4 & 0 & 1 & 2 & 4 & 4 & 4 & 4 \\
v_4 & 4 & 3 & 1 & 0 & 1 & 5 & 1 & 4 & 5 \\
v_5 & 1 & 1 & 2 & 1 & 0 & 1 & 3 & 3 & 3 \\
v_6 & 4 & 5 & 4 & 5 & 1 & 0 & 1 & 1 & 5 \\
v_7 & 3 & 4 & 4 & 1 & 3 & 1 & 0 & 1 & 4 \\
v_8 & 4 & 1 & 4 & 4 & 3 & 1 & 1 & 0 & 4 \\
\end{array}
$$

Table 1.1. Figure 1.1 shows the monophonic eccentricities of the graph $G$ vertices.

1.6. Note. In a tree $T$, there is only one path between any two vertices, $u$ and $v$, and so $d(u, v) = d_m(u, v) = D(u, v)$. Hence $\text{rad}_m T = \text{rad}_D T$ and $\text{diam}_m T = \text{diam}_D T$.

Table 2.1.2 lists the monophonic diameter and monophonic radius of a few common graphs.
Table 1.2. Several common graphs’ monophonic diameter and radius

<table>
<thead>
<tr>
<th>Graph</th>
<th>$K_p$</th>
<th>$C_p$</th>
<th>$W_{1,p-1}(p \geq 4)$</th>
<th>$K_{1,p-1}(p \geq 2)$</th>
<th>$K_{m,n}(m, n \geq 2)$</th>
<th>$P_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rad}_mG$</td>
<td>1</td>
<td>$p - 2$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\frac{n}{2}$</td>
</tr>
<tr>
<td>$\text{diam}_mG$</td>
<td>1</td>
<td>$p - 2$</td>
<td>$p - 3$</td>
<td>2</td>
<td>2</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>

1.7. **Theorem.** (a) If $a$, $b$ and $c$ are integers with $3 \leq a \leq b \leq c$, then a connected graph exists $G$ such that $\text{rad} \ G = a$, $\text{rad}_mG = b$ and $\text{rad}_D \ G = c$.

(b) If $a$, $b$ and $c$ are integers with $5 \leq a \leq b \leq c$, then a connected graph appears. $G$ exists in such a way that $\text{diam} \ G = a$, $\text{diam}_mG = b$ and $\text{diam}_D \ G = c$.

**Proof.** (a) Three examples are used to demonstrate the conclusion.

**Case (i)** $3 \leq a = b = c$. Consider $G = P_{2a+1}$. path, of order $2a + 1$. It is clear that $\text{rad} \ G = \text{rad}_mG = \text{rad}_D \ G = a$.

"**Case (ii)** $3 \leq a \leq b < c$. Let $F_1 : u_1, u_2, ..., u_{a-1}$ and $F_2 : v_1, v_2, ..., v_{a-1}$ be two copies of the path $P_{a-1}$ of order $a - 1$. Let $F_3 : w_1, w_2, ..., w_{b-a+3}$ and $F_4 : z_1, z_2, ..., z_{b-a+3}$ two duplicates of the path $P_{b-a+3}$ of order $b - a + 3$, and $F_5 = K_{c-b+1}$ the complete graph of order $c - b + 1$ with $V(F_5) = \{x_1, x_2, ..., x_{c-b+1}\}$. We construct the graph $G$ as follows: (i) identify the vertices $x_1$ in $F_5$ and $w_1$ in $F_3$; also identify the vertices $x_{c-b+1}$ in $F_5$ and $z_1$ in $F_4$; (ii) identify the vertices $w_{b-a+3}$ in $F_3$ and $u_2$ in $F_1$, and identify the vertices $z_{b-a+3}$ in $F_4$ and $v_2$ in $F_2$; and (iii) join each vertex $w_i (1 \leq i \leq b - a + 2)$ in $F_3$ and $u_1$ in $F_1$, and join each vertex $z_i (1 \leq i \leq c - b + 2)$ in $F_4$ and $v_1$ in $F_2$. The resulting graph $G$ is shown in Figure 1.2. It is easily verified that $e(v) = a$ if $v \in V(F_3)$; $e(v) = a$ if $v \in V(G) - V(F_5)$, $e_m(v) = b$ if $v \in V(F_5)$; $e_m(v) = b$ if $v \in V(G) - V(F_5)$ and $e_D(v) = c$ if $v \in V(F_5)$; and $e_D(v) = c$ if $v \in V(G) - V(F_5)$. It follows that $\text{rad} \ G = a$, $\text{rad}_mG = b$, and $\text{rad}_D \ G = c$.

"**Case (iii)** $3 \leq a < b = c$. Let $E_1 : v_1, v_2, ..., v_{2a+1}$ be a path of order $2a + 1$. Let $E_2 : u_1, u_2, ..., u_{b-a+3}$ and $E_3 : w_1, w_2, ..., w_{b-a+3}$ be two copies of the path $P_{b-a+3}$ of order $b - a + 3$, and let $E_i (4 \leq i \leq 2(b - a) + 3) = 2(b - a)$ copies of $K_1$. We construct the graph $G$ as follows: (i) identify the vertices $v_{a+1}$ in $E_1, u_1$ in $E_2$, and $w_1$ in $E_3$; (ii) identify the vertices $v_{a-1}$ in $E_1$ and $u_{b-a+3}$ in $E_2$, and identify the vertices $v_{a+3}$ in $E_1$ and $w_{b-a+3}$ in $E_3$; and (iii) join each $E_i (4 \leq i \leq b - a + 3)$ with $v_{a+1}$ in $E_1$ and $u_{i - 1}$ in $E_2$, and join each $E_i (b - a + 4 \leq i \leq 2(b - a) + 3)$ with $v_{a+1}$ in $E_1$ and $w_{i - b + a - 1}$ in $E_3$. The final graph, $G$, is displayed in Figure 1.3."
Case (i) $5 \leq a = b = c$. Let $G$ be a path of order $a + 1$. Then $diam G = diam_m G = diam_D G = a$.

Case (ii) $5 \leq a \leq b < c$ let $F_1 : u_1, u_2, ..., u_{a-1}$ be the path $P_a$ of order $a - 1$. $F_2 : v_1, v_2, ..., v_{b-a+3}$ be the path $P_{b-a+3}$ of order $b - a + 3$; and $F_3 : K_{c-b+1}$ be the full graph of order $c$. The graph $G$ is created as follows: (i) Identify the vertices $w_{b-a+3} \in F_2$ and $u_2 \in F_1$; (ii) join each vertex $w_i (1 \leq i \leq b - a + 2)$ in $F_2$ and $u_1$ in $F_1$; and (iii) identify the vertices $v_1$ in $F_3$ and $w_1$ in $F_2$; Figure 2.1.4 displays the final graph $G$.

It is easily verified that $e(v) = b$ if $v \in (V(F_2) - \{x_1\}) \cup \{u_{a-1}\}$, $e(v) < a$ if $v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$, and $e_m(v) = b$ if $v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $e_m(v) < b$ if $v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$, and $e_D(v) = c$ if $v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; and $e_D(v) < c$ if $v \in V(F_3) \cup (V(F_1) - \{u_{a-1}\})$. It follows that $diam G = a$, $diam_m G = b$, and $diam_D G = c$.

Case (iii)$^{**}$ $5 \leq a < b = c$. Let $E_i : v_1, v_2, ..., v_{a+1}$ be a path of order $a + 1$; $E_2 : w_1, w_2, ..., w_{b-a+3}$ be another path of order $b - a + 3$; and $E_i (3 \leq i \leq b - a + 2)$ be $b - a$ copies of $K_1$. Let $G$ be the graph obtained from $E_i$ for $i = 1, 2, ..., b - a + 2$ by identifying the vertices $v_{a-2}$ and $v_a$ of $E_1$ with $w_1$ and $w_{b-a+3}$ of $E_2$, respectively, and joining each $E_i (3 \leq i \leq b - a + 2)$ with $v_{a-2}$ and $w_1$. The graph $G$ is shown in Fig 1.5$^{**}$.
It is easily verified that \( e(v) = a \) if \( v \in \{v_1, v_{a+1}\} \); \( e(v) \leq a \) if \( v \in V(G) - \{v_1, v_{a+1}\} \), and \( e_m(v) = b \) if \( v \in \{v_1, v_{a+1}\} \); \( e_m(v) \leq b \) if \( v \in V(G) - \{v_1, v_{a+1}\} \), and \( e_p(v) = c \) if \( v \in \{v_1, v_{a+1}\} \); and \( e_D(v) \leq c \) if \( v \in V(G) - \{v_1, v_{a+1}\} \). It follows that \( rad G = a, rad_m G = b \) and \( rad_D G = c \). The inequality  \( rad G \leq diam G \leq 2 rad G \) and \( rad_D G \leq diam_D G \leq 2 rad_D G \) hold for any connected graph \( G \). This is not applicable to monophonic radius and monophonic diameter, though. For instance, it is clear that \( rad_m G = 1 \) and \( diam_m G = p - 3 \geq 3 \) so that \( diam_m G > 2 rad_m G \) when the graph \( G \) is the wheel \( W_{1,p-1} (p \geq 6) \). That there is a connected graph \( G \) with \( rad G = a \) and \( diam G = b \) if \( a \) and \( b \) are any two consecutive positive integers, then \( a \leq b \leq 2a \). That there is a connected graph \( G \) with \( rad_D G = a \) and \( diam_D G = b \) if \( a \) and \( b \) are any two consecutive positive integers, then \( a \leq b \leq 2a \).

The theorem that follows now provides a realization result for \( rad_m G \) and \( diam_m G \).

**1.8. Theorem.** There exists a connected graph \( G \) such that \( rad_m G = a \) and \( diam_m G = b \) if \( a \) and \( b \) are positive integers with \( a \leq b \).

**Proof.** Three cases are used to demonstrate this result.

**Case(i)** \( a = b \geq 1 \). Let \( G \) be the cycle \( C_{a+2} \). Then \( rad_m G = a \) and \( diam_m G = b \).

**Case (ii)** \( a < b \leq 2a \). Let \( C_1 : u_1, u_2, ..., u_{a+2}, u_1 \) be a cycle of order, and \( C_2 : v_1, v_2, ..., v_{b-a+2}, v_1 \) be a cycle of order, respectively. The graph that results from finding the vertex(es) \( u_1 \) of \( C_1 \) and \( v_1 \) of \( C_2 \) is denoted by \( G \). Since \( b \leq 2a, b - a + 2 \leq a + 2 \) follows naturally. It is obvious that for any \( x \in G, d_m(u_1, x) = a \) and \( d_m(u_1, u_{a+1}) = a \), and as a result, \( e_m(\{u_1\}) = a \). Furthermore, it is clear that \( rad_m G = a \) because no vertex in \( G \) has \( e_m(x) < a \), and \( e_m(u_3) = b \) because it is clear that \( d_m(u_3, v_3) = b \) and \( d_m(u_3, x) \leq b \). Additionally, it is clear that for each vertex \( x \in G, e_m(x) \leq b \), resulting in \( diam_m G = b \).

**Case (iii)** \( b > 2a \). Let \( G \) stand for the graph formed by finding the end vertex of the path, \( P_{2a} \), and the wheel’s center vertex, \( W = K_1 + C_{b+2} (b \geq 2) \). Since \( b > 2a \), each vertex of \( x \in V(C_{b+2}) \) because \( e_m(x) = b \). Additionally, every vertex of \( x \in V(P_{2a}) \) and \( e_m(v_a) = a \). As a result, \( rad_m G = a \) and \( diam_m G = b \).

**1.9. Remark.** For integers \( a \) and \( b \) with \( 2a < b \), each vertex in the graph \( G \) given in Fig 1.6, has monophonic eccentricity \( b \) or \( n(a \leq n \leq 2a) \). So, unlike standard eccentricity, if \( k \) is an integer such that \( rad_m G < k < diam_m G \), there may not be a vertex \( x \) of \( G \) such that \( e_m(x) = k \).
2. Monophonic center and monophonic periphery

2.1. Definition. The monophonic center $C_m(G)$ of $G$ is the subgraph that is generated by the $G$ single-note center vertices. If $e_m(v) = \text{rad}_m G$, a vertex $v$ in a connected graph $G$ is referred to as a monophonic central vertex. The monophonic periphery is the subgraph that $G$ monophonic peripheral vertices form.

2.2. Remark. It is not necessary for a connected graph's monophonic center to be connected. $C_m(G) = \{v_3, v_6\}$ in relation to the graph $G$ in Figure 2.1.

2.3. Theorem. Every graph has a connected monophonic center.

Proof. $G$ should be a graph. We demonstrate how $G$ is a graph's monophonic center. Let the monophonic diameter of $G$ be given by $l = d_m$. Let $P : u_1, u_2, \ldots, u_i$ and $Q : v_1, v_2, \ldots, v_i$ be two copies of the path $P_l$. By connecting each vertex of graph $G$ with $u_i$ in $P$ and $v_i$ in $Q$, the necessary graph $H$ shown in Figure 2.2 is obtained from graphs $G, P,$ and $Q$. Then, for each vertex $x$ in $G$, $e_{mH}(x) = d_m$, and for each vertex $x$ outside of $G$, $d_m + 1 \leq e_{mH}(x) \leq 2 d_m$. $C_m(H) = G$ because $V(G)$ is the collection of monophonic central vertices of $H$.

2.4. Theorem. Every connected graph's monophonic center $C_m(G)$. Some block of $G$ is a subgraph of $G$.

Proof. Assume a connected graph $G$ exists with a monophonic center $C_m(G)$ that is not a subgraph of a $G$ block. Afterward, $G$ has a cut vertex $v$, resulting in $G - v$ having two components, $H_1$ and $H_2$, each with $C_m(G)$ vertices. Let $u$ be a $G$ vertex such that $e_m(v) = d_m(u, v)$, and $P_1$ be the longest $u - v$ monophonic path in $G$. Consequently, at least one of $H_1$ and $H_2$ lacks $P_1$ vertices, for example, $H_2$. Now consider a vertex...
\( w \) in \( C_m(G) \) that belongs to \( H_2 \), and consider \( P_2 \) to be the longest \( v - w \) monophonic path in \( G \). \( P_1 \) followed by \( P_2 \) yields the \( u - w \) longest monophonic path with a length greater than \( P_1 \) because \( v \) is a cut vertex. This results in \( e_m(w) > e_m(v) \), implying the contradiction that \( w \) is not the monophonic central vertex of \( G \).

**2.5. Problem.** Considering any three positive integers \( a, b, \) and \( c \) with \( 1 \leq a \leq b \leq c \) whether a connected graph \( G \) exists \( \text{diam} G = a, \text{diam}_mG = b \) and \( \text{diam}_pG = c \)?

![Figure 2.2](image)

**Solution:** We consider the following four instances.

Case 1. \( a = 1 \). If such a graph exists, \( G \) is a complete graph of order \( c + 1 \) for some \( c \geq 1 \) because \( \text{diam} G = 1 \). Therefore, \( 1 = a = b \leq c \) and \( b = \text{diam}_mG = 1 \) and \( \text{diam}_pG = c \). For some \( c \geq 1 \), however, \( G \) is a complete graph of order \( c + 1 \) if \( a = b = 1 \), as a result, if and only if \( 1 = a = b \leq c \), there is a graph \( G \) with \( \text{diam} G = a = 1, \text{diam}_mG = b, \) and \( \text{diam}_pG = c \).

Case 2. \( a = b = c \).

A desired graph is one with a path of order \( c + 1 \). (In reality, \( \text{diam} T = \text{diam}_mT = \text{diam}_pT \) is a tree \( T \) property.)

Case 3. \( 2 \leq a \leq b < c \).

Let a path lead to the graph \( G, P : u_0, u_1, u_2, \ldots, u_c \) by joining the vertices \( u_c \) and \( u_t \) for \( a - 2 \leq t < c \), and \( u_i \) and \( u_j \) for \( b - 1 \leq i < j \leq c \) (avoiding the multiple edges formed during the construction). It is routine to check that \( \text{diam} G = a, \text{diam}_mG = b, \) and \( \text{diam}_pG = c \).

Case 4. \( 2 \leq a < b = c \).

First, suppose \( 2 \leq a \leq 3 \). Let \( P : u_0, u_1, u_2, \ldots, u_c \) be a monophonic path of length \( c \). Since \( a < c, P \) is not a \( u_0 - u_c \) geodesic. Let \( Q : u_0, v_1, v_2, \ldots, v_k, u_c \) be a \( u_0 - u_c \) geodesic. Since \( P \) is monophonic, \( v_1 = u_i \) for \( 2 \leq i \leq c \). Moreover \( v_1 = u_i \). Otherwise, \( P_1 : v_1, u_0, u_1, u_2, \ldots, u_c \) is a path of length \( c + 1 \), which is a contradiction. Similarly, we have \( v_k = u_{c - 1} \). By the same argument as above, we may assume that \( v_i = u_i \) for \( 1 \leq i \leq s \) or \( t \leq i \leq c - 1 \), where \( s < t \) and \( v_j = u_j \) for \( j = s + 1 \) or \( t - 1 \). Hence, \( d(u_0, u_c) \geq 4 \geq a + 1 \), which is a contradiction. Therefore, no such graphs exist in this subcase.

Let's say that's \( a \geq 4 \) now. We can create the graph \( G \) from the path \( P : u_0, u_1, u_2, \ldots, u_c \) by adding a new vertex \( v \) and connecting it to the vertices \( u_{c-a+3} \) and \( u_{c-2i-1} \) for \( 1 \leq 2i - 1 < c - a + 2 \). Verifying that \( \text{diam} G = a, \text{diam}_mG = b, \) and \( \text{diam}_pG = c \) is routine.

**2.6. Theorem.** "A non-trivial graph \( G \) is the monophonic periphery of some connected graph if and only if every vertex of \( G \) has monophonic eccentricity \( 1 \) or no vertex of \( G \) has monophonic eccentricity \( 1 \)."

**Proof.** "Suppose that every vertex of \( G \) has monophonic eccentricity \( 1 \). Then \( P_m(G) = G \). Next, suppose that no vertex of \( G \) has monophonic eccentricity \( 1 \). Hence for any vertex \( x \) in \( G \), there is a vertex \( y \) in \( G \) such that \( e_m(x) = d_m(x, y) \geq 2 \).

Clearly, \( e_m(x) \leq p - 1 \). Now, take \( p \) vertex disjoint paths \( P_i (1 \leq i \leq p) \) each of length \( p - 1 \) such that no vertex of \( P_i \) is a vertex of \( G \). Identify the end vertices of one path, say \( P_i \), with \( x \) and \( y \), thereby producing a cycle of length \( e_m(x) + p - 1 \). This is done for every vertex \( z = x \) of \( G \) by taking a
path $P_j(i = j)$. Let the graph obtained be $G_1$. Now, for every path $P_i(1 \leq i \leq p)$ in $G_1$, join each internal vertex of $P_i$ with every vertex of $V(G_1) - V(P_i)$, avoiding multiple edges. Let $H$ be the resulting graph obtained. (It is to be noted that if $y$ is a monophonic eccentric vertex of $x$, then $x$ is also a monophonic eccentric vertex of $y$, and adjoining a path as mentioned above, may or may not be done. This does not affect the monophonic eccentricity of any vertex in $H$.) Let $e_mH(v)$ denote the monophonic eccentricity of a vertex $v$ in $H$. Then it is clear that $e_mH(v) = p - 1$ for any vertex $v$ in $G$ and $e_mH(v) \leq p - 2$ for any vertex $v$ not in $G$. Hence $P_m(H) = G$. The graph in Fig. 9 shows the construction of the graph $H$ when $G$ is the path $v_1, v_2, v_3, v_4$, where $e_mH(v) = 3$ for every vertex $v$ in $G$ and $e_mH(v) = 2$ for every vertex $v$ not in $G$.

Conversely, let $G = P_m(H)$. Suppose that some but not all vertices of $G$ have monophonic eccentricity 1. Certainly $G$ is a proper subgraph of $H$. Therefore, for each vertex $x$ of $G$, it follows that $e_mH(x) = diam_m H \geq 2$. Let $u$ be a vertex of $G$ having monophonic eccentricity 1 in $H$. Then, $u$ is adjacent to all other vertices of $G$. Let $v$ be a vertex of $H$ such that $d_mH(u,v) = e_mH(u) = diam_m H \geq 2$. Hence $e_mH(v) = diam_m H$ and so $v \in P_m(H) = G$. Hence $u$ and $v$ are adjacent in $G$ and so $u$ and $v$ are also adjacent in $H$ such that $d_mH(u,v) = 1$, which is a contradiction”.

2.7. Definition. If $rad_mG = diam_m G$, or if $G$ is its own monophonic center, a connected graph $G$ is monophonic and self-centered.

3. Monophonic number of a graph

3.1. Definition. If each vertex $v$ of a graph $G$ lies on an $x - y$ monophonic path in $G$ for some $x, y \in S$, then the set $S$ of its vertices is said to be a monophonic set of $G$. The monophonic number is the minimum cardinality of a monophonic set of $G$.

and is indicated by of $G$. $m(G)$.

3.2. Example. The minimum monophonic sets of the graph $G$ shown in Figure 3.1 are $S_1 = \{x, w\}$ and $S_2 = \{u, w\}$, and as a result, $m(G) = 2$.

Figure 2.3.1 . A graph $G$ with $m(G) = 2$.

If a vertex $v$ in a graph $G$ is a member of each minimal monophonic set in $G$, then it is a monophonic vertex. Every vertex in $S$ is a monophonic vertex if $G$ has a singular minimal monophonic set $S$. In the following theorem, we demonstrate that a nontrivial linked graph $G$ has certain vertices that are monophonic $G$ vertices.

3.3. Theorem Every extreme vertex of a connected graph $G$ is contained in every monophonic set of the graph... Additionally, $S$ is the specific minimum monophonic set of $G$ if the set $S$ of all extreme vertices of $G$ is a monophonic set.

Proof. Let $S$ be a monophonic set of $G$ and let $u$ be an extreme vertex. Assume that $u \in S$. Then, for some $x, y \in S$, $u$ is an internal vertex of a $x - y$ monophonic path, let say $P$. Allow $v$ and $w$ to be $u$’s neighbors on $P$. This results in a contradiction because $v$ and $w$ are not contiguous and $u$ is not an extreme vertex. Consequently, $u$ is a member of every monophonic set of $G$. 

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3.4. **Corollary.** For complete graph $K_p(p \geq 2), m(K_p) = p$.

3.5. **Theorem.** Let $S$ be a monophonic set of $G$ and let $G$ be a connected graph with a cutvertex named $v$. Then, an element of $S$ is contained in each component of $G - v$.

**Proof.** Consider a component $B$ of $G - v$ which does not contain any vertex of $S$. Let any vertex in $B$ be $u$. Due to the fact that $S$ is a monophonic set, there is a pair of vertices $x$ and $y$ in $S$ such that $u$ lies in some $x - y$ monophonic path $P: x = u_0, u_1, u_2, \ldots, u_n = y$ in $G$ with $u \neq x, y, V$ being a cutvertex of both the $u - y$ subpath $P_2$ of $P$ and the $x - u$ subpath $P_1$ of $P$ contain $v$, it. Hence, which is a contradiction, $P$ is not a path.

3.6. **Theorem.** A connected graph $G$ cutvertex does not belong to any minimum monophonic set of $G$.

**Proof.** Let $S$ be the minimum monophonic set of $G$ and let $v$ be a cutvertex of $G$. Theorem 2.5 states that every part of $G - v$ contains a part of $S$. Let $U$ and $W$ be two separate parts of $G - v$, where $u \in U$ and $w \in W$. Following that, $v$ is an internal vertex of a monophonic path $u - w$. Let $S' = S - \{v\}$. Every vertex that is located on an $u - v$ monophonic path is evidently also Every vertex that is located on an $u - v$ monophonic path is evidently also monophonic set of $G$, which is in contrast to the statement that $S$ is a minimum monophonic set of $G$.

3.7. **Theorem.** If $G$ is a connected non-complete graph with a minimum cutset of vertices, then $m(G) \leq p - k$.

**Proof.** $G$ is an non-complete connected graph, hence it is obvious that $1 \leq k \leq p - 2$. Let $U$ be the minimum cutset of $G$, where $U = \{u_1, u_2, u_3, \ldots, u_k\}$. Let $S = V - U$ and $G_1, G_2, \ldots, G_r, (r \geq 2)$ be the parts of $G - U$. Then, for each $j (1 \leq j \leq r)$, every vertex $u_j (1 \leq i \leq k)$ is close to at least one vertex of $G_j$. Since $S$ is obviously a monophonic set of $G$, $m(G) \leq |S| = p - k$.

3.8. **Remark.** Theorem 2.3.7 has a sharp bound. For the cycle $C_4, m(C_4) = 2$. Also $\kappa = 2$ and $p - \kappa = 2$. Thus $m(G) = p - \kappa$.

3.9. **Theorem.** $G$ is complete If and only if $m(G) = p$, for any connected graph $G$ of order $p$.

**Proof.** Suppose $m(G) = p$. Assume that $G$ is not a fully complete graph. Then, there are two vertices $u$ and $v$ that are such that they are not next to one another in $G$. $G$ is connected, hence there is a monophonic path with length at least 2 from $u$ to $v$, let $x$ say $P$. In order for $x \neq u, v$, it must be a vertex of $P$. Therefore, $m(G) \leq p - 1$ is congruent since $S = V - \{x\}$ is a monophonic set of $G$.

3.10. **Definition:** Choose any vertex in $G$ to represent $x$. If any vertex $z$ with $d_m(x, y) < d_m(x, z)$, $z$ lies on an $x - y$ monophonic path, then vertex $y$ in $G$ is said to be an $x - z$ monophonic superior vertex.

3.11. **Theorem.** Let $x$ represent any $G$ vertex. Then, each $x$ - monophonic superior vertex is a monophonic eccentric vertex of $x$.

**Proof.** So that $e_m(x) = d_m(x, y)$, let $y$ be a monophonic eccentric vertex of $x$. There exists a vertex $z$ in $G$ such that $d_m(x, y) < d_m(x, z)$ and $z$ does not reside on any $x - y$ monophonic path, leading to the contradiction that $e_m(x) \geq d_m(x, z) > d_m(x, y)$, which occurs if $y$ is not an $x$ - monophonic superior vertex.

3.12. **Note.** Theorem 3.11 converse is untrue. The cycle $C_6$ has the following vertices: $v_1, v_2, v_3, v_4, v_5, v_6, v_1$, where $v_4$ is a $v_1$ - monophonic superior vertices and not a $v_1$ - monophonic eccentric vertices.

3.13. **Theorem.** Supposing $G$ is a connected graph, If and only if two vertices $x$ and $y$ exist, with $y$ being an $x$-monophonic superior vertex and every vertex of $G$ being on an $x - y$ monophonic path, then $m(G) = 2$.
Proof. Assume that $S = \{x, y\}$ is a minimum monophonic set of $G$ and that $m(G) = 2$. There is a vertex $z$ in $G$ with $d_m(x, y) < d_m(x, z)$ and $z$ does not reside on any $x - y$ monophonic path if $y$ is not an $x$-monophonic superior vertex. This results in a contradiction because $S$ is not a monophonic set of $G$.

4. Bounds for the monophonic number of a graph

We provide an improved upper bound for the monophonic number of a graph in the following theorem in terms of its order and monophonic diameter. We use the term "$d_m$" to represent the monoponic diameter diammG for convenience.

4.1. Theorem. $m(G) \leq p - d_m + 1$ if $G$ is a non-trivial connected graph with order $p$ and monophonic diameter $d_m$.

Proof. Let $P: u = v_0, v_1, \ldots, v_{d_m} = v$ be an $u - v$ monophonic path of length $d_m$. Let $u$ and $v$ be the vertices of $G$ such that $d_m(u, v) = d_m$. Let $S = V - \{v_1, v_2, \ldots, v_{d_m} - 1\}$. When $m(G) \leq |S| = p - d_m + 1$, it is evident that $S$ is a monophonic set of $G$. In order to ensure that the bound in Theorem 2.4.1 is sharp, for the complete graph $K_p (p \geq 2)$, $d_m = 1$ and $m(K_p) = p$.

4.2. Theorem. $2 \leq m(G) \leq g(G) \leq p$ for each connected graph $G$ of order $p$.

Proof. Every geodesic is a monophonic path, hence every geodesic set must also be a monophonic set. Consequently, $m(G) \leq g(G)$. The other disparities are trivial.

4.3. Remark. 3.1. Theorem 4.2 bounds are exact. Assuming that $K_p$ is a complete graph, $m(K_p) = g(K_p) = p$, $m(P_n) = g(P_n) = 2$ for the path $P_n$, which is non-trivial. Additionally, $m(G) = g(G)$ is a complete bipartite graph, an even cycle, or a non-trivial tree ($G$). In Theorem 4.2, every inequality is a rigorous inequality. $S = \{v_6, v_7, v_3\}$ is a minimum monophonic set of the graph $G$ shown in Figure 4.1 such that $m(G) = 3$ and no 3-elements subset of the vertex set is a geodesic set of $G$. A geodesic set of $G$ is $S \cup \{v_1\}$, hence it follows that $g(G) = 4$. As a result, we have $2 < m(G) < g(G) < p$.

![Figure 4.1. A graph G in Remark 4.3. with 2 < m(G) < g(G) < p](image)

References


