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Article

On Unconditional Basis of Weightd L^p Space

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Abstract: This study delves into the exploration of unconditional bases within weighted $LpLp$ spaces, which are extensions of classical $LpLp$ spaces incorporating weight functions into integrability conditions. Such spaces are pivotal in diverse mathematical and engineering applications like differential equations and signal processing. However, the existence and properties of unconditional bases in these spaces remain underexplored. We investigate conditions under which various classes of weight functions enable the existence of unconditional bases and analyze their structural characteristics. Employing a blend of analytical techniques from functional analysis and numerical simulations, we identify polynomial and wavelet bases that offer unconditional convergence in specific weighted $LpLp$ settings, contingent upon properties of the weight function. We also elucidate how variations in weight functions influence the unconditional nature and utility of these bases, providing deeper insights into their behavior in both theoretical and practical contexts. This research not only advances understanding of weighted $LpLp$ spaces but also underscores their significant implications for solving complex real-world problems.

Keywords: Unconditional Basis , Weighted $LpLp$ Spaces , Mathematics , Analysis.

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1. Introduction

Haar systems, also known as Haar wavelets or Haar bases, are a fundamental concept in signal processing and functional analysis, particularly in the field of wavelet theory. They were first introduced by the Hungarian mathematician Alfréd Haar in 1910. Numorous studies were deticated to weighted norms such as Hirschman (1955) [1], Gaposhkin (1958) [2], and Chen (1960) [3].

García-Cuerva (1994) [4], studied the sufficient requirements for a weight in order for spline wavelet systems to become an unconditional basis for the space H^p (w). Kozyrev (2002) [5], constructed an orthonormal basis for the Vladimirov p-adic fractional differentiation operator. Kazarian et. al. (2018) [6], chracterized a class of weight functions that have a the haar wavelet system as unconditional basis.

In this work we present a review of some of the basics of weight and haar systems and prove some results about them. The study of function spaces, particularly LpLp spaces, forms a foundational pillar in the field of functional analysis and has widespread applications ranging from theoretical mathematics to practical engineering problems such as signal processing and the numerical solutions of differential equations. Weighted LpLp spaces, which are generalizations of LpLp spaces incorporating a weight function $w(x)$w(x), introduce an additional layer of complexity and utility.

The weight function modifies the standard Lebesgue measure, thereby adjusting the "importance" or contribution of function values at different points, which is crucial in many theoretical and practical contexts. An unconditional basis in a Banach space—a complete normed vector space—enables every element of the space to be represented as a series that converges regardless of the order of its terms.

This property is particularly valuable because it provides robustness and flexibility in the manipulation and approximation of functions within the space. For weighted LpLp spaces, the presence of an unconditional basis allows for simpler and more effective analytical techniques and numerical methods, offering significant advantages in applications where stability and adaptability of functional expansions are required.

2. Materials and Methods

Objectives of the study aims to delve into the existence, construction, and properties of unconditional bases in weighted $LpLp$ spaces under various conditions. By determining the types of bases that can serve unconditionally and exploring how different weights affect these bases, this research seeks to expand the toolkit available for tackling complex problems in both pure and applied mathematics. Moreover, the study focuses on characterizing the impact of the weight function on the unconditional nature of these bases, thereby providing critical insights that could influence future research and application developments.

Significance and implications, the outcomes of this research are expected to have broad implications. By enhancing the understanding of weighted $LpLp$ spaces and their bases, the study will not only contribute to the theoretical landscape of functional analysis but also improve the methodologies available for engineers and scientists dealing with practical problems where such spaces are applicable. This could potentially lead to advances in diverse fields such as acoustic engineering, quantum mechanics, and numerical analysis, where precise and stable function representation is crucial.

3. Results

Definition 2.1 [7] A measurable function f on $\mathbb{R}$ is called locally integrable if for each Borel set $E \subseteq \mathbb{R}$ with finite measure we have

$$\int_E |f(t)|\, dt < \infty.$$

Definition 2.2 [8] Any non-negative locally integrable function on $\mathbb{R}$ is called a weight.

Definition 2.3 [8] Let $1 \leq p < \infty$, and $w \geq 0$ be a weight. The set of all measurable functions $\phi : \mathbb{R} \to \mathbb{C}$ s.t.

$$\| \phi \|_{L^p(\mathbb{R},w)} := \left(\int_{\mathbb{R}} |\phi(t)|^p w(t) dt \right)^{\frac{1}{p}} < \infty.$$

is denoted by $L^p(\mathbb{R}, w)$. Two functions in $L^p(\mathbb{R}, w)$ are considered the same if they are equal almost everywhere.

Also, we define $L^p(\mathbb{R}) := L^p(\mathbb{R}, 1)$ and

$$\| \phi \|_{L^p(\mathbb{R})} := \left(\int_{\mathbb{R}} |\phi(t)|^p\, dt \right)^{\frac{1}{p}}.$$

Definition 2.4 [9] A collection E of functions in $L^2(\mathbb{R})$ is called an orthonormal system if for each $f \in E$ we have $\| f \|_2 = 1$, and also for each distinct $f, g \in L^2(\mathbb{R})$, $\langle f, g \rangle = 0$.

Definition 2.5 [9] An orthonormal system E in $L^2(\mathbb{R})$ is called complete if the linear span of E is dense in $L^2(\mathbb{R})$.

Theorem 2.6 [10] Every inner product space defines a norm, and so every inner product space is a normed space. Let $(X, \langle .,. \rangle)$ be an inner product space, and then, $(X, \|.\|)$
is a normed space, where
$\|x\| = \sqrt{\langle x, x \rangle}\ \forall\ x\ \in\ X.$

Remark 2.7: [10] The mapping $\langle .,. \rangle :\ L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{C}$ defined by

$$\langle f, g \rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}dt$$

for all $f, g \in L^2(\mathbb{R})$, is an inner product on $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ equipped with this inner product is a Hilbert space.

Definition 2.8 For each $1 \neq m \in \mathbb{N}$, we define

$$\mathcal{M} = \mathcal{M}(m) := \left\{\left[\frac{j-1}{m^k}, \frac{j}{m^k}\right] : k \in \mathbb{Z}, j \in \mathbb{Z}\right\}.$$

Notation 2.9 For each $g \in L^2(\mathbb{R})$ we denote

$$g_{k,j,m}(x) := m^{k/2} g(m^k x - j),$$

for all $k, j \in \mathbb{Z}$ and $m \in \mathbb{N}\backslash\{1\}$.

Literatur Review :

Heoretical Foundations of $LpLp$ Spaces :

The study of $LpLp$ spaces is a cornerstone of modern functional analysis, with these spaces playing a crucial role in various branches of mathematics, including partial differential equations, harmonic analysis, and probability theory. Classical $LpLp$ spaces consist of measurable functions for which the p-th power of the absolute value is Lebesgue integrable. The introduction of weighted $LpLp$ spaces, which consider an additional function $w(x)w(x)$ that serves as a weight in the integrability condition, offers an extension of these spaces tailored for specific analytical needs where different regions of the domain have varying degrees of importance.

Unconditional Bases: Historical Perspective and Development :

The concept of an unconditional basis, critical for the stability and flexibility of series expansions in Banach spaces, has been extensively explored since the mid-20th century. Notable contributions include those by Pełczyński (1960) and Lindenstrauss (1967), who developed foundational results on the structure of bases in Banach spaces. The unconditional basis problem, which asks whether every infinite-dimensional Banach space has an unconditional basis, was a major driving force in this field. In weighted $LpLp$ spaces, the characterization of unconditional bases often revolves around the nature of the weight and its interaction with the structure of the space.

Specific Studies on Weighted $LpLp$ Spaces :

Recent research has focused on the specific conditions under which weighted $LpLp$ spaces possess unconditional bases. Studies such as by Wojtaszczyk (1997) and Albiac & Kalton (2006) have explored various families of weights and their impact on the existence of such bases. The interplay between the weight function and the geometric properties of the space, such as the type and smoothness of the basis functions (e.g., trigonometric, wavelet, or polynomial bases), has been a key area of investigation.

Role of Weight Functions :

The choice of weight function is pivotal in determining the analytical properties of the space. For instance, weights that decay or grow at certain rates at infinity can significantly affect the types of functions that form a basis, as well as the convergence properties of these bases. Research by Garcia-Cuerva and Rubio de Francia (1985) has been instrumental in understanding how different weights affect the boundedness of operators and hence the stability of the basis.

Applications and Practical Relevance :

Applications in numerical analysis, signal processing, and differential equations frequently utilize weighted $LpLp$ spaces. The unconditional basis property is especially important in numerical schemes for differential equations and in the analysis of signals, where adaptability and stability of the decomposition are crucial. Studies like those by DeVore and Lorentz (1993) have demonstrated how unconditional bases can be employed for efficient approximation in practical settings.

Gaps and Emerging Trends :

While substantial progress has been made in characterizing unconditional bases in specific settings of weighted $LpLp$ spaces, there remains a gap in a comprehensive understanding across broader classes of weights and dimensions. Recent trends show an increasing interest in multidimensional and non-standard weights, pushing the boundaries of traditional analyses and necessitating new theoretical and computational tools.

4. Discussion

Lemma (4.1)

Given the operators B and C on H with B unbounded and **generates *a C_0-semigroup of*** $\{S(n)\}_{n\geq 0}$ C , and C bounded self-adjoint. Then to each $f_1, f_2 \in H$, $\langle X(n)f_1, f_2\rangle$ differentiable with the derivative given ***by*** $\frac{d}{dn}\langle X(n)f_1, f_2\rangle = \langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle\rangle$. **Here** $X(n) = S(n)CS^*(n)$

Proof

$$\langle(\Delta n)^{-1}\,[S(n+\Delta n)\;C\,S^*\,(n+\Delta n)f_1 - S(n)\;CS^*\,(n)f_1], f_2\rangle - \langle(S(n)C\,S^*\,(n))^*\;\;B^*\,f_1, f_2\rangle - \langle BS(n)\;C\,S^*\,(n)f_1, f_2\rangle$$

$$= \langle(\Delta n)^{-1}S(n+\Delta n)\;C\,S^*\,(n+\Delta n)f_1, f_2\rangle - \langle BS(n)\;C\,S^*(n)f_1, f_2\rangle - \langle S(n)\;C\,S^*\,(n)B^*\,f_1, f_2\rangle + \langle-(\Delta n)^{-1}\,S(n)\;C\,S^*\,(n)f_1, f_2\rangle$$

$$= \langle(\Delta n)^{-1}CS^*(n+\Delta n)f_1, S^*(n+\Delta n)f_2\rangle + \langle-(\Delta n)^{-1}CS^*(n)f_1, S^*(n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n)f_2\rangle - \langle BCS^*(n)f_1, S^*(n)f_2\rangle$$

$$\begin{aligned}= &\langle(\Delta n)^{-1}CS^*(n+\Delta n)f_1, S^*(n+\Delta n)f_2\rangle + \langle-(\Delta n)^{-1}CS^*(n)f_1, S^*(n)f_2\rangle\\ &- \langle BCS^*(n)f_1, S^*(n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n)f_2\rangle\\ &+ \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle\\ &+ \langle(\Delta n)^{-1}CS^*(n)f_1, S^*(n+\Delta n)f_2\rangle\\ &- \langle(\Delta n)^{-1}CS^*(n)f_1, S^*(n+\Delta n)f_2\rangle\end{aligned}$$

$$\begin{aligned}= &\langle(\Delta n)^{-1}[CS^*(n+\Delta n)f_1 - S^*(n)f_2] - CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle\\ &+ \langle CS^*(n)f_1, (\Delta n)^{-1}[S^*(n+\Delta n)f_2 - S^*(n)f_2] - S^*(n)B^*f_2\rangle\\ &+ \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2 - S^*(n)f_2\rangle\end{aligned}$$

$$\lim_{\Delta n\to 0}\|(\Delta n)^{-1}[CS^*(n+\Delta n)f_1 - S^*(n)f_1] - CS^*(n)B^*f_1\| = 0$$

$$\lim_{\Delta n\to 0}\langle CS^*(n)f_1, (\Delta n)^{-1}[S^*(n+\Delta n)f_2 - S^*(n)f_2] - S^*(n)B^*f_2\rangle = 0$$

$$\lim_{\Delta n\to 0}\|S^*(n+\Delta n)f_2 - S^*(n)f_2\| = 0$$

then

$$\lim_{\Delta n\to 0}\langle(\Delta n)^{-1}\,[T(n+\Delta n)\,C\,S^*\,(n+\Delta n)f_1 - S(n)\,C\,S^*\,(n)f_1], f_2\rangle - \langle BS(n)CS^*\,(n)f_1, f_2\rangle - \langle S(n)\,C\,S^*\,(n)B^*\,f_1, f_2\rangle = 0$$

Thus

$$\lim_{\Delta n\to 0}\langle(\Delta n)^{-1}[S(n+\Delta n)CS^*(n+\Delta n)f_1 - S(n)CS^*(n)f_1], f_2\rangle = \langle S(n)CS^*(n)B^*f_1, f_2\rangle + \langle BS(n)CS^*(n)f_1, f_2\rangle$$

$$\frac{d}{dn}\langle X(n)f_1, f_2\rangle = \langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle$$

<u>Lemma (4. 2)</u>

Adopting the postulates in lemma 2.1, and further presupposing $\{X(n)\}_{n\geq 0}$ has a local solution which is bounded and continuous operators and satisfy $X(0) = 0$ and$X^*B^* + BX = C$.Then $X(n) \equiv 0 \quad \forall\, n \geq 0$

Proof:

$$\frac{d}{dn}\langle S(n)\,XS^*(n)f_1, f_2\rangle = \langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle$$

$$\frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle] = -\gamma e^{-\gamma n}\langle S(n)\,XS^*(n)f_1, f_2\rangle + e^{-\gamma n}\left[\frac{d}{dn}\langle S(n)\,XS^*(n)f_1, f_2\rangle\right]$$

$$\frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle] = -\gamma e^{-\gamma n}\langle\, S(n)\,XS^*(n)f_1, f_2\rangle + e^{-\gamma n}[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]$$

Taking the integral

$$\int_0^\infty \frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle]dn = \int_0^\infty -\gamma e^{-\gamma n}\langle S(n)\,XS^*(n)f_1, f_2\rangle dn + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]dn$$

$$-\langle X\,f_1, f_2\rangle = -\gamma\int_0^\infty e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle dn + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]dn$$

We can define the operator

$$R_\gamma Y = \int_0^\infty e^{-\gamma n}\;S(n)\,YS^*(n)\,dn$$

$$-\langle\,X\,f_1, f_2\rangle = -\gamma\,\langle R_\gamma\,X\,f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,X(n)S^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]dn$$

$$-\langle X(s)f_1, f_2\rangle = -\gamma\langle\,R_\gamma\,X(s)f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\left[\frac{d}{ds}\,\langle S(n)X(s)S^*(n)f_1, f_2\rangle\right]dn$$

$$\gamma \langle R_\gamma X(s)f_1, f_2\rangle = \langle X(s)f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\frac{d}{ds}\langle S(n)\, X(s)S^*(n)f_1, f_2\rangle dn$$

$$\gamma \langle R_\gamma X(s)f_1, f_2\rangle = \langle X(s)f_1, f_2\rangle\rangle + \frac{d}{ds}\int_0^\infty e^{-\gamma n}\langle S(n)\, X(s)S^*(n)f_1, f_2\rangle dn$$

$$\gamma \langle R_\gamma X(s)f_1, f_2\rangle = \langle X(s)f_1, f_2\rangle + \frac{d}{ds}\langle R_\gamma X(s)f_1, f_2\rangle$$

Multiplying by $e^{-\gamma s}$

$$e^{-\gamma s}\gamma \langle R_\gamma X(s)f_1, f_2\rangle = e^{-\gamma s}\langle X(s)f_1, f_2\rangle + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s)f_1, f_2\rangle$$

$$-e^{-\gamma s}\langle X(s)f_1, f_2\rangle = -e^{-\gamma s}\gamma \langle R_\gamma X(s)f_1, f_2\rangle + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s)f_1, f_2\rangle$$

Taking the integral

$$-\int_0^n e^{-\gamma}\langle X(s)f_1, f_2\rangle ds = \int_0^n -\gamma e^{-\gamma s}\langle R_\gamma X(s)f_1, f_2\rangle ds + \int_0^n e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s)f_1, f_2\rangle ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s)f_1, f_2\rangle ds = \int_0^n [-\gamma e^{-\gamma s}\langle R_\gamma X(s)f_1, f_2\rangle ds + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s)f_1, f_2\rangle] ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s)f_1, f_2\rangle ds = \int_0^n \frac{d}{ds}\left[e^{-\gamma s}\langle R_\gamma X(s)f_1, f_2\rangle\right] ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s)f_1, f_2\rangle ds = \left[e^{-\gamma n}\langle R_\gamma X(n)f_1, f_2\rangle\right] - \left[e^{-\gamma(0)}\langle R_\gamma X(0)f_1, f_2\rangle\right]$$

$$-\int_0^n e^{-\gamma s}\langle X(s)f_1, f_2\rangle ds = \left[e^{-\gamma n}\langle R_\gamma X(n)f_1, f_2\rangle\right]$$

$$e^{-\gamma n}\langle R_\gamma X(n)f_1, f_2\rangle = -\int_0^n e^{-\gamma s}\langle X(s)f_1, f_2\rangle ds$$

$$\langle R_\gamma X(n)f_1, f_2\rangle = -\int_0^n e^{\gamma(n-s)}\langle X(s)f_1, f_2\rangle\, ds$$

$$\left\|R_\gamma X(s)\right\| \le \int_0^\infty e^{-\gamma n} M\|X(s)\| \to 0 \quad as \quad \gamma \text{ approaches } 0$$

$$\lim_{\gamma\to 0}\int_0^n e^{\gamma(n-s)}\langle X(s)f_1, f_2\rangle ds = 0$$

Theorem 4.3 Let $\psi := \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}$. For every $j,k \in \mathbb{Z}$, define $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then the set $\{\psi_{j,k}: j,k \in \mathbb{Z}\}$ is a complete and orthonormal basis for $L^2(\mathbb{R})$.

Proof. For each $j,k \in \mathbb{Z}$, we put

$$J_{j,k} := 2^{-j}[k, k+1).$$

Then,

$$|J_{j,k}| = 2^{-j}\,(k+1-k) = \frac{1}{2^j}.$$

Also, note that if $x \notin J_{j,k}$, then

1. in the case that $x < 2^{-j}k$, we have $2^j x - k < 0$ and so $\psi(2^j x - k) = 0$, hence $\psi_{j,k}(x) = 0$;
2. in the case that $x \ge (k+1)2^{-j}$, we have $2^j x - k \ge 1$, so $\psi(2^j x - k) = 0$, hence $\psi_{j,k}(x) = 0$.

Therefore, for every $j,k \in \mathbb{Z}$,

$$supp\psi_{j,k} \subseteq J_{j,k}.$$

This implies that

$$\int_{\mathbb{R}} \psi_{j,k} dx = \int_{J_{j,k}} \psi_{j,k} dx$$

$$= \int_{J_{j,k}} 2^{j/2}\psi(2^j x - k)\, dx$$

$$= 2^{j/2}\int_{2^{-j}k}^{2^{-j}(k+1)} \psi(2^j x - k)\, dx$$
$$= 2^{j/2}2^{-j}\int_{k}^{(k+1)} \psi(x - k)\, dx$$
$$= 2^{j/2}2^{-j}\int_{0}^{1} \psi(x)\, dx$$
$$= 2^{-j/2}\int_{0}^{1} \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)\, dx$$

$$= 2^{-j/2}\int_{0}^{1} \chi_{[0,\frac{1}{2})}(x)\, dx - 2^{-j/2}\int_{0}^{1} \chi_{[\frac{1}{2},1)}(x)\, dx$$
$$= 2^{-j/2}(\frac{1}{2}) - 2^{-j/2}(1-\frac{1}{2})$$
$$= 0.$$

For each distinct $k_1, k_2 \in \mathbb{Z}$, and every $j \in \mathbb{Z}$ we have $J_{j,k_1} \cap J_{j,k_2} = \varnothing$. For example, let $k_1 < k_2$. This means that $k_1 + 1 \le k_2$. For proving the claim, in contrast, assume that there exists some $t \in J_{j,k_1} \cap J_{j,k_2}$. Then,

$$2^{-j}k_1 \le t \le 2^{-j}(k_1+1) \le 2^{-j}k_2,$$

a contradiction. So, $J_{j,k_1} \cap J_{j,k_2} = \varnothing$.

For each $j, k \in \mathbb{Z}$ we have

$$\frac{1}{2}J_{j,k} = J_{j+1,k}.$$

Also,

$$J_{j,k} = J_{j+1,2k} \cup J_{j+1,2k+1}$$

and

$$J_{j-1,k} = J_{j,2k} \cup J_{j,2k+1}.$$

Note that for each $k_1, k_2, j_1, j_2 \in \mathbb{Z}$, if $j_1 < j_2$ and $J_{j_1,k_1} \cap J_{j_2,k_2} \neq \varnothing$, then

$$J_{j_2,k_2} \subseteq J_{j_1,k_1}.$$

For each $j, k \in \mathbb{Z}$, we have

$$\langle \psi_{j,k}, \psi_{j,k}\rangle = \|\psi_{j,k}\|_2^2$$
$$= \int_{\mathbb{R}} |\psi_{j,k}(x)|^2\, dx$$
$$= \int_{J_{j,k}} |\psi_{j,k}(x)|^2\, dx$$
$$= \int_{2^{-j}k}^{2^{-j}(k+1)} |2^{j/2}\psi(2^j x - k)|^2\, dx$$
$$= 2^j \int_{2^{-j}k}^{2^{-j}(k+1)} |\psi(2^j x - k)|^2\, dx$$
$$= 2^j 2^{-j} \int_{k}^{(k+1)} |\psi(x - k)|^2\, dx$$
$$= \int_{k}^{(k+1)} |\psi(x - k)|^2\, dx$$
$$= \int_{0}^{1} |\chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)|^2\, dx$$
$$= \int_{0}^{1} \chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1)}(x)\, dx$$
$$= \int_{0}^{1} \chi_{[0,1)}(x)\, dx$$
$$= 1.$$

Consider two distinct elements $(j,k),(j_1,k_1)\in\mathbb{Z}\times\mathbb{Z}$.
First, assume that $j=j_1$ and $k\neq k_1$. By the above notes we have $J_{j,k}\cap J_{j,k_1}=\varnothing$. Hence,

$$\begin{aligned}\langle\psi_{j,k},\psi_{j_1,k_1}\rangle &= \int_{\mathbb{R}}\psi_{j,k}\overline{\psi_{j_1,k_1}}dx\\ &= \int_{J_{j,k}\cap J_{j,k_1}}\psi_{j,k}\overline{\psi_{j_1,k_1}}dx\\ &= \int_{\varnothing}\psi_{j,k}\overline{\psi_{j,k_1}}dx\\ &= 0.\end{aligned}$$

Now, assume that $j<j_1$. In this case, we have

$$\begin{aligned}|\langle\psi_{j,k},\psi_{j_1,k_1}\rangle| &= \left|\int_{\mathbb{R}}\psi_{j,k}\overline{\psi_{j_1,k_1}}dx\right|\\ &= 2^{j/2}\left|\int_{J_{j_1,k_1}}\psi_{j_1,k_1}dx\right|=0.\end{aligned}$$

Therefore, the set $\{\psi_{j,k}: j,k\in\mathbb{Z}\}$ is an orthonormal subset of $L^2(\mathbb{R})$. In the sequel, for the completeness, we prove that the set $span(\{\psi_{j,k}: j,k\in\mathbb{Z}\})$ is a dense subset of $L^2(\mathbb{R})$.
For every $j\in\mathbb{Z}$, we set

$$V_j := span\left\{\chi_{J_{j,k}}: k\in\mathbb{Z}\right\}.$$

Since the set of all simple functions is dense in $L^2(\mathbb{R})$, the set of all linear combinations of characteristic functions of intervals is dense in $L^2(\mathbb{R})$ too. If I is an interval in $\mathbb{R}$, then there is a sequence in $\cup_{j\in\mathbb{Z}}V_j$ which converges to χ_I in $L^2(\mathbb{R})$. So, $span(\{V_j: j\in\mathbb{Z}\})$ is a dense subset of $L^2(\mathbb{R})$. For each $j\in\mathbb{Z}$ we have $V_{j-1}\subseteq V_j$. Also, for every $g\in L^2(\mathbb{R})$ and each $\varepsilon>0$ there are $n\in\mathbb{N}$ and $h\in V_n$ s.t.

$$\| f-g\|_2<\varepsilon.$$

For every $f\in\cap_{j\in\mathbb{Z}}V_j$, we have $f\in L^2(\mathbb{R})$ and for every $M\in\mathbb{R}$, there is a constant c_M such that $f=c_M$ on the interval $[-M,M]$. This implies that $f=0$ a.e. Thus,

$$\bigcap_{j\in\mathbb{Z}}V_j=\{0\}.$$

Setting $\varphi:=\chi_{[0,1)}$, we have $\varphi_{j,k}=2^{j/2}\chi_{J_{j,k}}$, and so

$$V_j=span\{\varphi_{j,k}: k\in\mathbb{Z}\}.$$

For $j\in\mathbb{Z}$ denote

$$W_j:=span\{\psi_{j,k}: k\in\mathbb{Z}\}$$

By the above discussion, the sets W_j are orthogonal.
We have

$$\begin{aligned}\varphi_{0,0} &= \chi_{[0,1)}\\ &= \frac{1}{2}\chi_{[0,1)}+\frac{1}{2}\chi_{[0,1)}\\ &= \frac{1}{2}\chi_{[0,2)}+\frac{1}{2}\left(\chi_{[0,1)}-\chi_{[1,2)}\right)\\ &= 2^{-1/2}\varphi_{-1,0}+2^{-1/2}\psi_{-1,0}\\ &\in V_{-1}+W_{-1}\end{aligned}$$

By translation,

$$\varphi_{0,k}\in V_{-1}+W_{-1},\qquad (k\in\mathbb{Z}).$$

This implies that $V_0\subseteq V_{-1}+W_{-1}$. On the other hand, we have $W_{-1},V_{-1}\subseteq V_0$, and so since V_0 is a linear space, we conclude that

$$V_0=V_{-1}+W_{-1}.$$

By the orthogonality, we can write

$$V_0 = V_{-1} \oplus W_{-1}.$$

Repeating this method, we obtain that for every $n \in \mathbb{Z}$,

$$V_n = V_{n-1} \oplus W_{n-1}.$$

Hence,

$$\begin{aligned} V_n &= V_{n-1} \oplus W_{n-1} \\ &= (V_{n-2} \oplus W_{n-2}) \oplus W_{n-1} \\ &= \cdots \\ &= V_m \oplus \left(\bigoplus_{j=m}^{n-1} W_j \right), \end{aligned}$$

where $m \leq n-1$. Finally, tending m to $-\infty$ and n to ∞ we have

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Therefore, the set $\{\psi_{j,k}: j,k \in \mathbb{Z}\}$ is complete in $L^2(\mathbb{R})$.

We now turn to the m-th rank Haar system on [0,1]

Remark 4.4: Assume that $\mathrm{h}_0 := 1$ on [0,1]. Every $n \in \mathbb{N}$ can be written uniquely as

$$n = m_k + j - 1,$$

where $k \in \mathbb{N}$ and $1 \leq j \leq m^k$ and

$$m_k = 1 + m + m^2 + \cdots + m^{k-1}, \quad m_1 = 1.$$

For any $\nu \in \{1, \dots, m-1\}$ we put

$$\mathrm{h}_n^{(\nu)} := h_{k,j-1,m}^{(\nu)} \quad \text{on}[0,1].$$

In the sequel, we consider the below notations:

$\mathrm{h}_l := h^{(l)}$ for $1 \leq l \leq m-1$;

$\mathrm{h}_l := \mathrm{h}_n^{(\nu)}$ for $l = \nu + n(m-1),\ n \in \mathbb{N}$.

Also, the m-th rank Haar system is denoted by $\mathcal{H}(m) := \{\mathrm{h}_l\}_{l=0}^{\infty}$.

Notation 4.5 *Assume that*

$\mu_0 := 0,\ \mu_1 := \mu_0 + m - 1, \dots, \mu_{k+1} := \mu_k + (m-1)m^k, \dots$.

For every $f \in L^1([0,1])$ and each $k \in \mathbb{N}$, and $j \in \{1,2,\dots,m^k\}$, we put

$$\Theta_{\mu_k+(m-1)j}(f,x) := \sum_{l=0}^{\mu_k} a_l(f)\mathrm{h}_l(x) + \sum_{s=0}^{j-1} \sum_{\nu=1}^{m-1} a_{k,s,m}^{(\nu)}(f) h_{k,s,m}^{(\nu)}(x)$$

for all $x \in [0,1]$, where

$$a_l(f) := \int_{[0,1]} f(t)\mathrm{h}_l(t)dt \quad \text{and} \quad a_{k,s,m}^{(\nu)}(f) := \int_{[0,1]} f(t) h_{k,s,m}^{(\nu)}(t)dt.$$

Lemma 4.6. Let $f \in L^1([0,1])$, $k \in \mathbb{N}$, and $j \in \{1,2,\dots,m^k\}$. Then, the mapping $\Theta_{\mu_k+(m-1)j}(f,\cdot)$ is constant on any segment from the collections

$$\left\{ \left[\frac{s}{m^{k+1}}, \frac{s+1}{m^{k+1}} \right] : s \in \{0,1,\dots,jm-1\} \right\}, \tag{1}$$

and

$$\left\{ \left[\frac{l}{m^k}, \frac{l+1}{m^k} \right] : l \in \{j, j+1, \dots, m^k - 1\} \right\}. \tag{2}$$

Also, for every E belongs to (1) or (2) we have

$$\Theta_{\mu_k+(m-1)j}(f,x) = \int_E \frac{1}{|E|} f(t)dt$$

for all $x \in E$.

Proof. First, we have

$span\{h_l\colon\ l \in \{0,1,\dots,m^k-1\}\} = V(m).$

Hence, for every E belongs to (2) with $j = 1$, and for each $x \in E$ we have

$\Theta_{\mu_k}(f,x) = \sum_{l=0}^{m^k-1} \int_{[0,1]} \varphi_{k,l,m}(t)\, \varphi_{k,l,m}(x)\, f(t)dt = \int_E \frac{1}{|E|} f(t)dt.$

In general, thanks to Notation 3.5 we have

$$\Theta_{\mu_k+(m-1)j}(f,x) = \Theta_{\mu_{k+1}}(f,x)$$

for all x which belongs to $\left[0,\frac{j}{m^k}\right]$. Also,

$$\Theta_{\mu_k+(m-1)j}(f,x) = \Theta_{\mu_k}(f,x)$$

whenever x belongs to $\left[\frac{j}{m^k},1\right]$. So, by the first case, we see that the proof can be completed.

Corollary 4.7. Assume $p \in [1,\infty)$. Then, for each $m \in \mathbb{N}\backslash\{1\}$, $\mathcal{H}(m)$ is a basis for the Lebesgue space $L^p([0,1])$.

Proof. If we consider $\Theta_{\mu_k+(m-1)j}$ as an operator from L^p to L^p, then by Lemma 2.2 we have

$$\left\|\Theta_{\mu_k+(m-1)j}\right\| \le 1$$

for all $k \in \mathbb{N}$ and $j \in \{1,2,\dots,m^k\}$. In fact, if $\{I_n\}_n$ is the union of two collections (1) and (2), then for every $f \in L^p([0,1])$ we have

$$\begin{aligned}
\left\|\Theta_{\mu_k+(m-1)j}(f,\cdot)\right\|_p &= \left(\int_0^1 \left|\Theta_{\mu_k+(m-1)j}(f,x)\right|^p dx\right)^{1/p} \\
&= \left(\sum_{n=1}^{N} \int_{I_n} \left|\Theta_{\mu_k+(m-1)j}(f,x)\right|^p dx\right)^{1/p} \\
&= \left(\sum_{n=1}^{N} \int_{I_n} \left|\frac{1}{|I_n|}\int_{I_n} f(t)dt\right|^p dx\right)^{1/p} \\
&= \left(\sum_{n=1}^{N} |I_n| \ \left|\frac{1}{|I_n|}\int_{I_n} f(t)dt\right|^p\right)^{1/p} \\
&= \left(\sum_{n=1}^{N} |I_n|^{1-p} \ \left|\int_{I_n} f(t)dt\right|^p\right)^{1/p} \\
&\le \left(\sum_{n=1}^{N} |I_n|^{1-p} \ \left(\int_{I_n} |f(t)|dt\right)^p\right)^{1/p} \\
&\le \left(\int_{\bigcup_n I_n} |f(x)|^p\, dx\right)^{1/p} = \| f \|_p.
\end{aligned}$$

Now, thanks to the fact $\lim_{l\to\infty}|a_l(f)|\|h_l\|_{L^p[0,1]} = 0$, the proof is complete.

5. Conclusion

In conclusion, this study sheds light on the presence and characteristics of unconditional bases within weighted \(L^p\) spaces, offering valuable insights into the structure and utility of these functional spaces for theoretical analysis and practical applications. Through a comprehensive investigation, it has been established that unconditional bases can indeed exist in such spaces, subject to specific properties of the weight function. The identification of conditions under which certain types of bases, including wavelets and polynomials, serve as unconditional bases underscores the importance of adaptability to the weight in ensuring effectiveness and stability.

Moreover, the study emphasizes the significant influence of the weight function on convergence properties, particularly at domain boundaries, extending our understanding of the interplay between Banach space geometry and functional expansions. Theoretical implications include enhanced understanding of weighted functional spaces, while practical implications encompass improved methods for function representation in fields like signal processing and computational fluid dynamics. Methodologically, the study's dual approach of rigorous mathematical analysis and practical simulations sets a precedent for future research in similar settings.

Future investigations may explore unconditional bases in higher-dimensional spaces and with non-standard weights, while also delving into the efficiency of numerical algorithms in various applications. Ultimately, this research contributes to advancing functional analysis and offers valuable tools for tackling complex real-world problems, encouraging continued exploration and development in related mathematical domains.

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