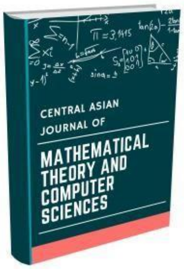

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Article

Optimal Solution of Laplace's Equation Using Finite Differences

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Abstract: In this study, we address the numerical approximation of Laplace's equation, a fundamental partial differential equation in physics and engineering, using finite difference methods. Specifically, we explore the application of the standard five-point and diagonal five-point methods to solve the equation under given boundary conditions. A grid network is established in the first quadrant of the coordinate plane, with values at grid points determined by either the standard five-point, diagonal five-point, or a combined approach. The resulting system of equations is formulated as a matrix and solved using the Gaussian method to obtain the values at each grid node. Our findings demonstrate the effectiveness of these methods in accurately approximating solutions to Laplace's equation, with potential implications for improving computational techniques in related fields.

Keywords: Laplace's Equation, Finite Difference Methods, Five-point Method, Gaussian Method, Numerical Approximation

1. Introduction

Many natural phenomena can be expressed by differential equations, some of which can be solved by classical methods and others by using approximate methods to solve them. Differential equations are involved in all fields such as physics, chemistry, mathematics and computer science[1]. Differential equations have been studied by many scientists including Lorenz, Rabinovich-Fabrikant, and others[2-4]. Most areas of applied mathematics, including heat flow, mechanics, fluid flow, and optics, can be explained by partial differential equations [5]. Analytical methods can only be used to solve a small number of these equations. Numerical techniques make it easy to develop approximate solutions[6].

Among all the numerical methods available for solving partial differential equations, the finite difference approach is most frequently employed. It replaces the derivatives in the equations and boundary conditions with their estimated finite differences. After that, the provided equation is changed into a system of linear equations that may be resolved iteratively. [7-8]. In this study, partial differential equations are solved numerically in order to determine an approximate solution.

Mathematics has advanced remarkably along with the advancement of science, especially in the field of solving differential equations. Differential equations have been solved using a variety of techniques, such as the finite difference approach, the finite

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element approach, and the finite volume approach[9-11]. One of the earliest and most basic approaches to solving differential equations is the finite difference technique of derivatives. Euler made the discovery of it in one spatial dimension in 1968. Then, in 1908, Runge refined and expanded it to the second dimension. Finite difference approaches were first applied numerically in the early 1950s[12]. Finite differences have been used to solve Poisson's equation, Laplace's equation, the heat conduction equation, the wave equation, and other partial differential equations, producing theoretical conclusions on their validity, convergence, and stability[13–14].

2. Materials and Methods

Partial differential equations are solved using the finite difference method. cutting the first quarter of the coordinate plane into a square or rectangle Drawings and lines parallel to the x- and y-axes that cross one another to form a network known as intersection points or nodes are displayed in Figure (1)(a, b).

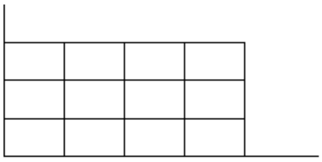

Figure 1. A. Nodes in a rectangle

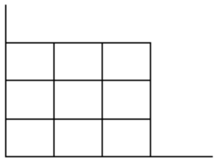

Figure 2. B. nodes in a square

The network of lines is known by

$x_i = in, \quad i = 1,2,3,\ldots, \quad (1)\,, \quad y_i = im, \quad i = 1,2,3,\ldots \quad (2)$

If m=n, then the network is regular, and we use the symbol($A_{i,j}$) to represent the numerical solution at the point(x_i, y_j).

replacing the differential equations' partial derivatives at the nodes with the proper difference approximations. This procedure, known as estimating the partial differential equation, involves approximating the partial differential equations (1), (2) using finite difference equations at each network point (node).

$$(A_x)_{i,j} = \frac{1}{2n}\left(A_{i+1,j} + A_{i-1,j}\right) \quad (3)\ , \quad \left(A_y\right)_{i,j} = \frac{1}{2m}\left(A_{i,j+1} + A_{i,j-1}\right) \quad (4)$$

By differentiating equations (3,4) the second derivative

$$(A_{xx})_{i,j} = \frac{1}{n^2}\left(A_{i+1,j} - 2A_{i,j} + A_{i-1,j}\right) \quad (5)$$

$$(A_{yy})_{i,j} = \frac{1}{m^2}\left(A_{i,j+1} - 2A_{i,j} + A_{i,j-1}\right) \quad (6)$$

Five-point standard technique

The five-point formula for equations (5,6) at$(\,i, j)$ is found by using the four neighborhoods $((i+1, j), (i-1, j), (i, j+1), (i, j-1))$, as seen in Figure (2-a).

Diagonals Five-point formulas

The four neighborhoods can be used to get the five extra points on diagonal lines.

$((i+1, j+1),\ (i-1, j+1),\ (i+1, j-1),\ (i-1, j-1))$ as depicted in the figure(2-b)

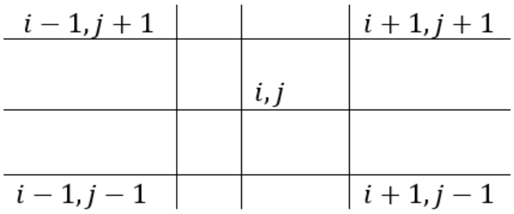

Figure 2. A

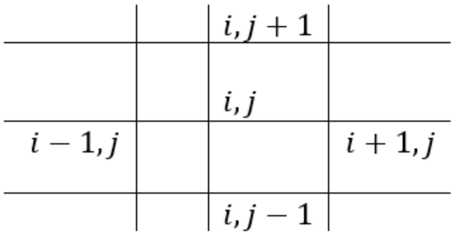

Figure 2. B

3. Results and Discussion

Solution of Laplace's equation

Laplace's equation can be solved using both the standard and diagonal five-point formulas.

Three approaches exist for resolving Laplace's equation.

Apply the standard five-point formula to solve Laplace's equation.

Laplace's equation

$A_{xx} + A_{yy} = \nabla^2 v = 0$ With the boundary conditions

$A(x, y) = f(x, y)$

From equations (5,6) we get

$$A_{xx} + A_{yy} = \frac{1}{n^2}\left(A_{i+1,j} - 2A_{i,j} + A_{i-1,j}\right) +$$

$$(7)\frac{1}{m^2}\left(A_{i,j+1} - 2A_{i,j} + A_{i,j-1}\right) = 0$$

By multiplying both sides of the equation by (n^2) and assuming that $h = \frac{n}{m}$, we conclude that

$$\left(A_{i+1,j} - 2A_{i,j} + A_{i-1,j}\right) + h^2\left(A_{i,j+1} - 2A_{i,j} + A_{i,j-1}\right) = 0 \quad (8)$$

If $n = m$ then $h = 1$

$$A_{i+1,j} + A_{i-1,j} + A_{i,j+1} + A_{i,j-1} - 4A_{i,j} = 0 \quad (9)$$

The conventional five-point formula for solving the Laplace equation is represented by these approximations.

The way in which equation (9) can be expressed is

$$A_{i,j} = \frac{1}{4}\left[A_{i+1,j} + A_{i-1,j} + A_{i,j+1} + A_{i,j-1}\right] \quad (10)$$

We observe that the value of$(A_{i,j})$ is the average of four points' values along the x-axes and y-axes

Apply the diagonal five-point formula to solve Laplace's equation.

To determine the diagonal five-point formula for the variables (x, y), we employ the Taylor series.

$$A_{i+1,j+1} = A_{i,j} + h\frac{\partial y}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 v}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 v}{\partial x^3} + \cdots \quad (11)$$

$$A_{i-1,j-1} = A_{i,j} - h\frac{\partial y}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 v}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 v}{\partial x^3} + \cdots \quad (12)$$

From the equations (11),(12) we get

$$A_{i+1,j+1} + A_{i-1,j-1} = 2A_{i,j} + h^2\frac{\partial^2 v}{\partial x^2} \quad \Rightarrow A_{xx} = \frac{1}{h^2}\left[A_{i+1,j+1} + A_{i-1,j-1} - 2A_{i,j}\right] \quad (13)$$

$$A_{i+1,j-1} = A_{i,j} + h\frac{\partial y}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 v}{\partial y^2} + \frac{h^3}{3!}\frac{\partial^3 v}{\partial y^3} + \cdots \quad (14)$$

$$A_{i-1,j+1} = A_{i,j} - h\frac{\partial y}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 v}{\partial y^2} - \frac{h^3}{3!}\frac{\partial^3 v}{\partial y^3} + \cdots \quad (15)$$

From the equations (11),(12) we get

$$A_{i+1,j-1} + A_{i-1,j+1} = 2A_{i,j} + h^2\frac{\partial^2 v}{\partial y^2} \quad \Rightarrow \quad A_{yy} = \frac{1}{h^2}\left[A_{i+1,j-1} + A_{i-1,j+1} - 2A_{i,j}\right] \quad (16)$$

From the equations (13),(16) we get on the Laplace's equation

$$A_{xx} + A_{yy} = \frac{1}{2h^2}\left[A_{i+1,j+1} + A_{i-1,j-1} + A_{i+1,j-1} + A_{i-1,j+1} - 4A_{i,j}\right]$$

Since the Laplace's equation is $A_{xx} + A_{yy} = 0$

$$\frac{1}{2h^2}\left[A_{i+1,j+1} + A_{i-1,j-1} + A_{i+1,j-1} + A_{i-1,j+1} - 4A_{i,j}\right] = 0$$

$$A_{i,j} = \frac{1}{4}\left[A_{i+1,j+1} + A_{i-1,j-1} + A_{i+1,j-1} + A_{i-1,j+1}\right] \quad (17)$$

The diagonal five point formula is the name given to this method of solving Laplace's equation.

Use the conventional and diagonal five-point formulas to solve Laplace's equation:

We wish to solve Laplace's equation which is A_xx+A_yy=0

Assume that (R) is a square area that can be split into a network of tiny side-h squares. A (x,y) values on the boundary (C) should be supplied as)Ci), and both mesh and boundary points should be shown in the picture.

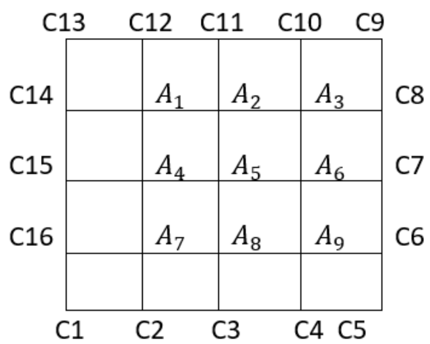

Figure 3. Obtain them using the diagonal five-points formula

Since the values of $(A_1, A_3, A_5, A_7, A_9)$ are located on the major diagonal, as shown in Figure (3), we may obtain them using the diagonal five-points formula (17).

$A_5 = \frac{1}{4}[C_1 + C_5 + C_9 + C_{13}]$, $A_1 = \frac{1}{4}[A_5 + C_{11} + C_{13} + C_{15}]$

$v_3 = \frac{1}{4}[C_7 + C_9 + C_{11} + A_5]$, $A_7 = \frac{1}{4}[C_3 + A_5 + C_{15} + C_1]$

$A_9 = \frac{1}{4}[C_5 + C_7 + v_5 + C_3]$

Applying the conventional five-point method (10), we determine a value of (A_2, A_4, A_6, A_8).

$$A_2 = \frac{1}{4}[A_3 + C_{11} + A_1 + A_5], \qquad A_4 = \frac{1}{4}[A_5 + A_1 + C_{15} + A_7]$$
$$A_6 = \frac{1}{4}[C_7 + A_3 + A_5 + A_9], \qquad A_8 = \frac{1}{4}[A_9 + A_5 + A_7 + C_3]$$

Solve the system of equations:

To solve the system of equations, we could use the direct technique or the Gauss method. Boundary conditions and the approximation locations of the differences at each node are used. The resulting system is an algebraic equation system of the type Av=d.

As seen in figure(3), the network's nodes are numbered in an ordered fashion

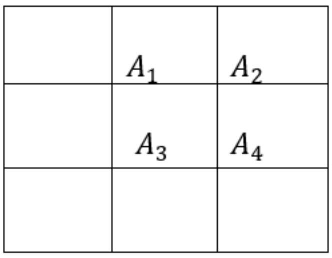

Figure 4

$At(1)$: $A_2 + A_3 - 4A_1 = d_1$
$At(2)$: $A_1 + A_4 - 4A_2 = d_2$
$At(3)$: $A_1 + A_4 - 4A_3 = d_3$
$At(4)$: $A_2 + A_3 - 4A_4 = d_4$
where (d_3, d_3, d_3, d_3) are the result of the node boundaries.
Let's calculate the approximation error for Laplace's equation.
Application of Poisson and Laplace equation solving

Determine the Laplace equation's solution $A_{xx} + A_{yy} = 0$ utilizing the conventional five-point formula in the given area R, according to the specified boundary constraints

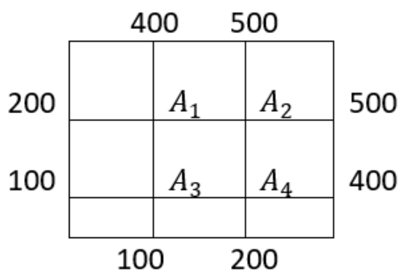

Figure 5

We note that there is symmetry in both the boundary conditions and the partial differential equation.

about (A_1, A_4), meaning that$A_1 = A_4$, and that the boundary conditions are asymmetric about A_2, A_3.. We have to figure out $A_1, A_2, A_3$and A_4.

$$A_1 = \frac{1}{4}[A_2 + 400 + 200 + A_3] \quad \Rightarrow \quad 4A_1 - A_2 - A_3 = 600 \qquad (1)$$
$$A_2 = \frac{1}{4}[500 + 500 + A_1 + A_4] \quad \Rightarrow \quad -A_1 + 4A_2 - A_4 = 1000 \qquad (2)$$
$$A_3 = \frac{1}{4}[A_4 + A_1 + 100 + 100] \quad \Rightarrow \quad -A_1 + 4A_3 - A_4 = 200 \qquad (3)$$
$$A_4 = \frac{1}{4}[400 + A_2 + A_3 + 200] \quad \Rightarrow \quad -A_2 - A_3 + 4A_4 = 600 \qquad (4)$$

Gauss elimination is the approach we use to solve the system of equations. We make use of the matrix.

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 600 \\ -1 & 4 & 0 & -1 & 1000 \\ -1 & 0 & 4 & -1 & 200 \\ 0 & -1 & -1 & 4 & 600 \end{array}\right];\ \frac{R_1}{-2},\ \left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ -1 & 4 & 0 & -1 & 1000 \\ -1 & 0 & 4 & -1 & 200 \\ 0 & -1 & -1 & 4 & 600 \end{array}\right],R_2+R_1, R_3+R_1,$$

$$\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ 0 & 15/4 & -1/4 & -1 & 1150 \\ 0 & -1/4 & 15/4 & -1 & 350 \\ 0 & -1 & -1 & 4 & 600 \end{array}\right],\frac{4}{15}R_2,\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ 0 & 1 & -1/15 & -4/15 & 920/3 \\ 0 & -1/4 & 15/4 & -1 & 350 \\ 0 & -1 & -1 & 4 & 600 \end{array}\right], \begin{array}{c} R_3+\frac{1}{4}R_2 \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ 0 & 1 & -1/15 & -4/15 & 920/3 \\ 0 & 0 & 224/15 & -16/15 & 1280/3 \\ 0 & 0 & -16/15 & 56/15 & 2720/3 \end{array}\right],\frac{15}{224}R_3,\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ 0 & 1 & -1/15 & -4/15 & 920/3 \\ 0 & 0 & 1 & -1/14 & 200/7 \\ 0 & 0 & -16/15 & 56/15 & 2720/3 \end{array}\right],$$

$$R_4+\frac{16}{15}R_3\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 150 \\ 0 & 1 & -1/15 & -4/15 & 920/3 \\ 0 & 0 & 1 & -1/14 & 200/7 \\ 0 & 0 & 0 & 384/105 & 6560/7 \end{array}\right]$$

From the fourth equation we get: $\frac{384}{105}A_4=\frac{6605}{7} \quad \Rightarrow \quad A_4=\frac{1025}{4}=256.25,$

From the third equation we get: $A_3=\frac{39}{384}+\frac{1}{14}A_4 \quad \Rightarrow \quad A_3=\frac{375}{8}=46.875$

From the second equation we get: $A_2=\frac{920}{3}+\frac{1}{15}A_3+\frac{4}{15}A_4 \quad \Rightarrow \quad A_2=\frac{3025}{8}=378.125$

$$A_1=150+\frac{1}{4}A_2+\frac{1}{4}A_3 \quad \Rightarrow \quad A_1=\frac{1025}{4}=256.25$$

2) use the boundary values indicated in the diagram to solve Laplace's equation for the following square mesh.

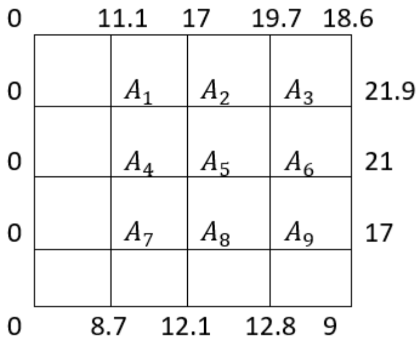

Figure 5

Now we shall compute A_1, A_3, A_5, A_7, A_9 by the diagonal five-points formula.

$$A_{i,j}=\frac{1}{4}\left[A_{i+1,j+1}+A_{i-1,j-1}+A_{i+1,j-1}+A_{i-1,j+1}\right]$$

$$A_5=\frac{1}{4}[9+0+18.6+0]=6.9$$

$$A_1=\frac{1}{4}[A_5+17+0+0]=5.975, \qquad A_3=\frac{1}{4}[21+18.6+17+A_5]=15.875$$

$$A_7=\frac{1}{4}[21.1+A_5+0+0]=4.75, \qquad A_9=\frac{1}{4}[9+21+A_5+21.1]=12.25$$

Now we shall compute A_2, A_4, A_6, A_8 by the standard five point formula

$$A_{i,j}=\frac{1}{4}\left[A_{i+1,j}+A_{i-1,j}+A_{i,j+1}+A_{i,j-1}\right]$$

$$A_2=\frac{1}{4}[A_3+17+A_1+A_5]=11.4375, \qquad A_4=\frac{1}{4}[A_5+A_1+0+A_7]=4.40625$$

$$A_6=\frac{1}{4}[21+A_3+A_5+A_9]=14.00625, \qquad A_8=\frac{1}{4}[A_9+A_5+A_7+21.1]=9$$

4. Conclusion

There are several finite difference methods, including (the standard five-point method, the five-point diameter method, the Schmidt method, and the Amplist method) to solve partial differential equations, including (Laplace's equation, Poisson's equation, the heat conduction equation, and the wave equation). In this research, we will solve Laplace's equation using the traditional five-point formula and the five-point diameter formula, which is regarded as one of the finite difference approaches. First, a network is formed in the first quarter of the coordinate plane by drawing lines parallel to the two axes. The intersection points of these lines are called nodes or network points, where the standard five-point formulas and their diameters are used to find the values of these nodes. There are several ways to solve Laplace's equation and find the nodes

1. Use the standard five-point formula using the four squares (1) and find equations, then solve these equations using the Gauss method and find the values of the variables that represent the nodes
2. Use the diagonal five-point formula using the four squares (1) to find the values of the nodes
3. Use the standard five-point formulas and their diameters, where the values of the nodes that lie on the diameters are found using the five-point diameter formula and the values of the rest of the nodes are found using the standard five-point formula

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