CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES
https://cajmtcs.centralasianstudies.org/index.php/CAJMTCS
Volume: 06 Issue: 01 | January 2025 ISSN: 2660-5309

Article

Finding The Norm View of The Optimal Formula of The Quadratury Error Function in Gilbert Space

Mamatova Zilolakhon Khabibullokhonovna[1], Hakimjonova Sarvinoz Iqboljon qizi[2]

1. Associate Professor, Doctor of Philosophy in Pedagogical Sciences, PhD, Fergana State University, Fergana, Uzbekistan
2. Graduate Student, Fergana State University, Fergana, Uzbekistan

* Correspondence: mamatova.zilolakhon@gmail.com, sarvinozmuhammadjonova921@gmail.com

Abstract: The implementation of quadrature formulas serves as an essential tool for numerical integration because they convert definite integrals into discrete summation approximations. Numerical analysis depends on the Euler-Maclaurin quadrature formula for wide scientific use since it enables computational error determination and correction work. The application of quadrature formulas within Hilbert space needs optimal configurations because they reduce approximation errors. These mathematical formulas have been applied in numerous spaces by previous researchers although additional development is required to elevate both computational speed and accuracy levels. Existing research on optimal quadrature formulas fails to establish the specific norm of error functional within Hilbert space using an extended Euler-Maclaurin formula. The current need exists to develop an optimal formulation that reduces errors successfully yet keeps mathematical precision intact. The study constructs an optimal quadrature formula in Hilber t space through determination of the error functional norm using Riesz's theorem and extremum function theory. The study establishes these optimized coefficients to achieve precision improvements in numerical quadrature computations. A new improved quadrature formula emerges from this research which satisfies Sard's problem definitions. Through explicit and mathematical derivations the norm of the error functional becomes clear as it demonstrates achieved minimum error bounds. The research introduces a mathematically founded quadrature formula specialized for Hilbert spaces to achieve error reduction in integration methods. The implementation of Riesz's theorem together with extremum function theory delivers a new method to construct error functionals. New mathematical discoveries from the study enhance numerical integ ration techniques thus improving the precision of computational mathematics and physical as well as engineering applications' quadrature formulas. The proposed integration formula provides substantial improvements for minimizing errors so it enhances theoretical computational and applied research results.

Keywords: Quadrature formula, Euler- Maclaurin formula, Hilbert formula.

Citation: Khabibullokhonovna. M. Z. Finding The Norm View of The Optimal Formula of The Quadratury Error Function in Gilbert Space. Central Asian Journal of Mathematical Theory and Computer Sciences 2025, 6(1), 128-134.

Received: 18th Feb 2025
Revised: 20th Feb 2025
Accepted: 25th Feb 2025
Published: 27th Feb 2025

1. Introduction

The Euler–Maclaurin quadrature formula is used to relate discrete sums to the definite integral. [1] This formula is used in numerical calculations, in the development of quadrature formulas, and in various fields from acoustics to quantum physics. The Euler–Maclaurin formula is used in numerical integration , acoustic environment analysis , optimal quadrature formulas , and numerical calculations .

The Euler-Maclaurin formula is written as follows:

$$\sum_{k=a}^{b} f(k) \approx \int_a^b f(x)dx + \frac{f(b)+f(a)}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} (f^{(2j-1)}(b) - f^{(2j-1)}(a)) + R_m$$

Here, are B_{2j} the Bernoulli [2] numbers R_m and are the remainder terms.

In this article, we will discuss $W_2^{(6,5)}$ the Gilbert space new An extended version of the Euler-Maclaurin quadrature formula was constructed and the form of the norm of the error functional was found using Riesz's theorem and the extremum function for linear functionals.[3]

Literature analysis

Optimal quadrature formula error in Gilbert space functional norm appearance find for of the matter to be put and his/her to the solution until calculations [4] take will go and in this Kh.M. Shadimetov , AR Hayatov , Optimal quadrature formulas with positive coefficients in $L_2^{(m)}(0,1)$ space, J. Comput . Appl. In math books cited from formulas is used .

2. Materials and Methods

Optimal quadrature formula error in Gilbert space functional norm appearance find for given spaces research was done and this for given of space private empty seeing It was released . [5]

3. Results

1. The problem.We are the following quadrature the formula let's see :

$$\int_0^1 \varphi(x)\, dx \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) + \sum_{\beta=0}^{N} C_1[\beta]\varphi(h\beta) + $$
$$+ \sum_{\beta=0}^{N} C_3[\beta]\varphi'''(h\beta)dx + \sum_{\beta=0}^{N} C_5[\beta]\varphi^{(5)}(h\beta) \quad (1)$$

in this C_0, C_1, C_3- known coefficients . That is

$$C_0[\beta] = h/2, \beta = 0$$
$$C_0[\beta] = h, \beta = \overline{1, N-1}$$
$$C_0[\beta] = h/2, \beta = N \qquad C_1[\beta] = h^2/12, \beta = 0$$
$$C_1[\beta] = 0, \beta = \overline{1, N-1}$$
$$C_1[\beta] = -h^2/12, \beta = N \qquad C_2[\beta] = h^4/720, \beta = 0$$
$$C_2[\beta] = 0, \beta = \overline{1, N-1}$$
$$C_2[\beta] = -h^4/720, \beta = N$$

$C_5[\beta]$ -(1) square the formula for now unknown coefficients ,$h = \frac{1}{N}$, N −natural number.

Here, the integral is φ functions $W_2^{(6,5)^*}(0,1)$ into space relevant . In this $W_2^{(6,5)}$ space as fifth orderly product absolute uninterrupted sixth orderly product $L_2(0,1)$ into space [6] relevant functions class we get.This $W_2^{(6,5)^*}$ (0,1) in class scalar multiplication as follows included :

$$<\varphi, \psi> = \int_0^1 (\varphi^{(6)}(x) + \varphi^{(5)}(x))*(\psi^{(6)}(x) + \psi^{(5)}(x))dx \quad (2)$$

half scalar to multiply relative Gilbert space This is a scalar . multiplication using following norm let's see :

$$\|\varphi|W_2^{(6,5)}\|| = (\int_0^1 (\varphi^{(6)}(x) + \varphi^{(5)}(x))dx)^{1/2} \quad (3)$$

(1) quadrature formula that it was a mistake

$$\int_0^1 \varphi(x)dx - (\sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) + \sum_{\beta=0}^{N} C_1[\beta]\varphi'(h\beta) +$$
$$+ \sum_{\beta=0}^{N} C_3[\beta]\varphi'''(h\beta) + \sum_{\beta=0}^{N} C_5[\beta]\varphi^{(5)}(h\beta)) \quad (4)$$

to separate It is said.[7] This to separate $W_2^{(6,5)}(0,1)$ in space determined following functional suitable comes :

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x - h\beta) + \sum_{\beta=0}^{N} C_1[\beta]\delta'(x - h\beta) +$$

$$+ \sum_{\beta=0}^{N} C_3[\beta]\delta^{(3)}(x-h\beta) + \sum_{\beta=0}^{N} C_5[\beta]\delta^{(5)}(x-h\beta) \quad (5)$$

where $\varepsilon_{[0,1]}(x)$ - is the characteristic function of the [0,1] section, $\delta(x)$ - is the Dirac delta function, i.e. $\delta(x)$ -is also functional, and its effect on a smooth function, i.e. its value, is defined as follows.[8]

$(\delta(x),\varphi(x)) = \varphi(0)$, $(\delta(x-a),\varphi(x)) = \varphi(a)$, $(\delta^{(\alpha)}(x-a),\varphi(x)) = (-1)^{\alpha}\cdot\varphi^{(\alpha)}(a)$ **Proof** : Here $\ell(x)$ functional $\varphi(x)$ in function value as follows is determined

$$(\ell(x),\varphi(x)) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx. \quad (6)$$

Based on equation (6), taking into account (5), it is indeed the case that the differential $\ell(x)$ error functional (4) $\varphi(x)$ We make sure that the value in is ,[9]that is

$$(\ell,\varphi) = \int_{-\infty}^{\infty} \ell(x)\cdot\varphi(x)dx = (\int_{-\infty}^{\infty} \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-h\beta) +$$

$$+ \sum_{\beta=0}^{N} C_1[\beta]\delta'(x-h\beta) + \sum_{\beta=0}^{N} C_3[\beta]\delta^{(3)}(x-h\beta) + \sum_{\beta=0}^{N} C_5[\beta]\delta^{(5)}(x-h\beta))\cdot\varphi(x)dx =$$

$$= \int_0^1 \varphi(x)dx - (\sum_{\beta=0}^{N} C_0[\beta]\int_{-\infty}^{\infty} \delta(x-h\beta)\cdot\varphi(x)dx + \sum_{\beta=0}^{N} C_1[\beta]\int_{-\infty}^{\infty} \delta'(x-h\beta)\cdot\varphi(x)dx +$$

$$+ \sum_{\beta=0}^{N} C_3[\beta]\int_{-\infty}^{\infty} \delta^{(3)}(x-h\beta)\cdot\varphi(x)dx + \sum_{\beta=0}^{N} C_5[\beta]\delta^{(5)}(x-h\beta)\cdot\varphi(x)dx =$$

$$= \int_0^1 \varphi(x)dx - (\sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) - \sum_{\beta=0}^{N} C_1[\beta]\varphi'(h\beta)$$

$$+ \sum_{\beta=0}^{N} C_5[\beta]\varphi'''(h\beta) + \sum_{\beta=0}^{N} C_5[\beta]\varphi^{(5)}(h\beta))$$

So, the above from[10] the result It seems we made a mistake in (4). as follows writing we get possible

$$(\ell,\varphi) = \int_0^1 \varphi(x)dx - (\sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) +$$

$$+ \sum_{\beta=0}^{N} C_1[\beta]\varphi'(h\beta) + \sum_{\beta=0}^{N} C_3[\beta]\varphi'''(h\beta) + \sum_{\beta=0}^{N} C_5[\beta]\varphi^{(5)}(h\beta)).$$

(1) quadruture formula ℓ error functional $W_2^{(6,5)*}(0,1)$ joint into space relevant linear is functional . This with together , (5) error functional $W_2^{(6,5)}(0,1)$ in space determined for following equalities satisfaction is a condition .

$$(\ell,1) = 1 - \sum_{\beta=0}^{N} C_0[\beta] = 0, \quad (7)$$

$$(\ell,x) = \frac{1}{2} - \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta) = 0, \quad (8)$$

$$(\ell,x^2) = \frac{1}{3} - \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta)^2 - 2\sum_{\beta=0}^{N} C_1[\beta](h\beta) = 0, \quad (9)$$

$$(\ell,x^3) = \frac{1}{4} - \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta)^3 - 3\sum_{\beta=0}^{N} C_1[\beta](h\beta)^2 - 6\sum_{\beta=0}^{N} C_3[\beta] = 0, \quad (10)$$

$$(\ell,x^4) = \frac{1}{5} - \sum_{\beta=0}^{N} C_0[\beta]\varphi(h\beta)^4 - 4\sum_{\beta=0}^{N} C_1[\beta](h\beta)^3 - 24\sum_{\beta=0}^{N} C_3[\beta](h\beta) = 0, \quad (11)$$

$$(\ell,e^{-x}) = 1 - e^{-1} - \sum_{\beta=0}^{N} C_0[\beta]\cdot e^{-(h\beta)} + \sum_{\beta=0}^{N} C_1[\beta]\cdot e^{-(h\beta)} +$$

$$+ \sum_{\beta=0}^{N} C_3[\beta]\cdot e^{-(h\beta)} + \sum_{\beta=0}^{N} C_5[\beta]\cdot e^{-(h\beta)}. \quad (12)$$

From the above equations (7)-(11), the fourth term of the quadrature formula (1) is orderly to many accuracy means . We $C_5[\beta]$have chosen the coefficients such that conditions (7)-(11) are satisfied. We need to find the currently unknown $C_5[\beta]$, $\beta = 0,1,..,N$coefficients such that they satisfy equality (12). So we $C_5[\beta]$have only condition (12) for the coefficients.[11]

(1) The quadrature formula (6) $W_2^{(6,5)*}(0,1)$ is a linear functional of the error space, where - is the joint space of $W_2^{(6,5)*}$this $W_2^{(6,5)}$space. Then, from the Cauchy-Schwarz inequality (6), the absolute value of the error is estimated from above as

$$|(\ell,\varphi)| \leq \|\varphi|W_2^{(6,5)}(0,1)\| \cdot \|\ell|W_2^{(6,5)*}(0,1)\|.$$

From here we have (1) quadrature formula (6) error $\ell(x)$ error functional

$$\left\|\ell\,|W_2^{(6,5)*}(0,1)\right\| = \sup_{\varphi,\|\varphi\|\neq 0} \frac{|(\ell,\varphi)|}{\left\|\varphi\,|W_2^{(6,5)}(0,1)\right\|}$$

the form n . It is not difficult to see here $\ell(x)$ that the norm of the error functional depends on the coefficients.$C_5[\beta], \beta = \overline{0,N}$ Naturally , all one functions space $W_2^{(6,5)}(0,1)$ in the elements (1) quadrature formula error high border the most small value find important is considered . The problem of finding the minimum of the norm of the error functional $C_5[\beta]$ with respect to the coefficients is Sard. issue will be .[12]

The quadrature formula obtained as a result of solving this problem is optimal in the sense of Sard. is called the quadrature formula .

This of the work main purpose $W_2^{(6,5)}(0,1)$ in space (1) for the quadrature formula Sardis issue from undressing consists of , that is

$$\left\|\overset{\circ}{\ell}\right\| = \inf_{C_5[\beta]} \|\ell\| \tag{13}$$

satisfying $C_5[\beta]$ the equality coefficients from finding consists of .

Thus, $W_2^{(6,5)}(0,1)$ to construct an optimal quadrature formula in the Sard sense in the form (1) in the space, we need to solve the following two problems in sequence.[13]

1 – issue. (1) quadrature formula (5) error functional norm calculation

Problem 2. Find the coefficients satisfying equality (13) $C_5[\beta]$.

We are below first by solving the problem Let's do it .

2. Calculate the norm of the error functional.

$\ell(x)$ To calculate the norm of the error functional, we $\psi_\ell$use an extremal function that satisfies the following equality

$$(\ell,\psi_\ell) = \|\ell|W_2^{(6,5)*}(0,1)\| \cdot \|\psi_\ell|W_2^{(6,5)}(0,1)\|$$

It should also be noted that for $W_2^{(6,5)}(0,1)$ any ℓlinear functional defined in a Hilbert space ψ_ℓ , an extremum function was found in [1] and it was shown that the extremum function is a solution to the following boundary value problem

$$\psi_\ell^{(2m)}(x) - \psi_\ell^{(2m-2)}(x) = (-1)^m \ell(x) \tag{14}$$

$$(\psi_\ell^{(m+s)}(x) - \psi_\ell^{(m+s-2)}(x))|_{x=0}^{x=1} = 0,\ s = 0,1,\ldots, m-1, \tag{15}$$

$$(\psi_\ell^{(m)}(x) - \psi_\ell^{(m-1)}(x))|_{x=0}^{x=1} = 0 \tag{16}$$

also $\psi_\ell$for an extremal function following theorem proven :

Theorem 2.1 [17]. *(14) – (16) – the solution to the boundary value problem is an extremal function $\ell$for the error functional $\psi_\ell$and it has the following form*

$$\psi_\ell(x) = (-1)^m \ell(x) * G_m(x) + P_{m-2}(x) + de^{-x},$$

in this

$$G_m(x) = \frac{\mathrm{sign}x}{2}\left(\frac{e^x - e^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!}\right),$$

$P_{m-2}(x)$ *- an arbitrary* $m-2$*degree polynomial, dan arbitrary real number .*

Furthermore, the following equality is shown to be valid . These equalities linear functionals for Riss from the theorem come came out

$$\|\ell|W_2^{(m,m-1)*}\| = \|\psi_\ell|W_2^{(m,m-1)*}\| \text{and } (\ell,\psi_\ell) = \|\ell|W_2^{(m,m-1)*}\|^2. \tag{17}$$

Now, we $W_2^{(6,5)}(0,1)$ can use the theorem above to find the extremum function for the error functional $m = 6$in the space we are considering.ℓ We can do it without it.

Thus, from Theorem 2.1, we obtain the extremum function $m = 6$for the error functional ψ_ℓ defined in the case space ℓ, $W_2^{(6,5)}(0,1)$ which is as follows:

$$\psi_\ell(x) = \ell(x) * G_6(x) + P_4(x) + de^{-x}, \tag{18}$$

in this $P_4(x)$ fourth level polygamy and

$$G_6(x) = \frac{sgn\, x}{2} \cdot \left(\frac{e^x - e^{-x}}{2} - \frac{x^9}{9!} - \frac{x^7}{7!} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right) \tag{19}$$

For the solution to the first problem following appropriate .

Theorem 2.2. *(1) quadrature formula (5) norm of the error functional for following expression appropriate*

$$\begin{aligned}\|\ell\|^2 = (\ell, \psi_\ell) = &-\sum_{\beta=0}^N \sum_{\gamma=0}^N C_5[\beta]C_5[\gamma]G_1(h\beta - h\gamma) + \\ &+2\sum_{\beta=0}^N C_5[\beta]\left(\int_0^1 G_4'(x-h\beta)dx + \sum_{\gamma=0}^N C_0[\gamma]G_4'(h\beta - h\gamma) - \sum_{\gamma=0}^N C_1[\gamma]G_3(h\beta - h\gamma) - \right. \\ &\left.-\sum_{\gamma=0}^N C_3[\gamma]G_2(h\beta - h\gamma)\right) + \sum_{\beta=0}^N \sum_{\gamma=0}^N C_0[\beta]C_0[\gamma]G_6(h\beta - h\gamma) - \\ &-2\sum_{\beta=0}^N \sum_{\gamma=0}^N C_0[\beta]C_1[\gamma]G_6'(h\beta - h\gamma) - 2\sum_{\beta=0}^N \sum_{\gamma=0}^N C_0[\beta]C_3[\gamma]G_5'(h\beta - h\gamma) + \\ &-\sum_{\beta=0}^N \sum_{\gamma=0}^N C_1[\beta]C_1[\gamma]G_5(h\beta - h\gamma) - 2\sum_{\beta=0}^N \sum_{\gamma=0}^N C_1[\beta]C_3[\gamma]G_4(h\beta - h\gamma) - \\ &-\sum_{\beta=0}^N \sum_{\gamma=0}^N C_3[\beta]C_3[\gamma]G_3(h\beta - h\gamma) - 2\sum_{\beta=0}^N C_0[\beta]\int_0^1 G_6(x - h\beta)dx + \\ &+2\sum_{\beta=0}^N C_1[\beta]\int_0^1 G_6'(x - h\beta)dx + 2\sum_{\beta=0}^N C_3[\beta]\int_0^1 G_5'(x - h\beta)dx + \int_0^1\int_0^1 G_6(x - y)dxdy.\end{aligned} \tag{20}$$

This on the ground

$$G_1(x) = \frac{\text{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2}\right), \qquad G_2(x) = \frac{\text{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - x\right),$$

$$G_3(x) = \frac{\text{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^3}{3!} - x\right), \qquad G_4(x) = \frac{\text{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right),$$

$$G_5(x) = \frac{\text{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^7}{7!} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right). \tag{21}$$

Proof . (20) is expressed by equality (17).$m = 6$ when, (5) and (18) equalities We will take it into account.

So ,

$$\|\ell\|^2 = (\ell, \psi_\ell) = (\ell, \ell(x) \cdot G_6(x) + P_4 + de^{-x}) = (\ell, \ell(x) \cdot G_6) + P_4 + d \cdot e^{-x}$$

From there, taking into account equalities (7)- (11) , we get the following let's go

$$\|\ell\|^2 = (l, \psi_6) = (\ell, \ell(x) \cdot G_6(x)) \tag{22}$$

to calculate $\ell * G$, we first $\|\ell\|^2$ calculate . To the following we have :

$$(\ell \cdot G_6)(x) = \int_{-\infty}^{\infty} \ell(y) \cdot G_6(x - y)dy =$$

$$= \int_0^1 G_6(y - x)dy - \sum_{\beta=0}^N C_0[\beta] \cdot G_6(h\beta - x) - \sum_{\beta=0}^N C_1[\beta] \cdot G'_6(h\beta - x) -$$

$$-\sum_{\beta=0}^N C_3[\beta] \cdot G_6^{(3)}(h\beta - x) - \sum_{\beta=0}^N C_3[\beta] \cdot G_6^{(5)}(h\beta - x) =$$

$$= \int_0^1 G_6(x - y)dy - \sum_{\beta=0}^N C_0[\beta] \cdot G_6(x - h\beta) + \sum_{\beta=0}^N C_1[\beta] \cdot G'_6(x - h\beta) +$$

$$+\sum_{\beta=0}^N C_3[\beta] \cdot G_6^{(3)}(x - h\beta) + \sum_{\beta=0}^N C_3[\beta] \cdot G_6^{(5)}(x - h\beta).$$

now last PCB in consideration take (1 8) from the expression $\|\ell\|^2$ for the following we get :

$$\|\ell\|^2 = (\ell, \psi_\ell) = (\ell, \ell + G_6) = \int_{-\infty}^{\infty} \ell(x) \cdot (\ell \cdot G_6)(x)dx$$

Above expression on known simplifications done increasing , (21) equalities and their derivatives in consideration take (20) PCB harvest Theorem proved .

The first issue has been resolved . [14]

4. Discussion

This study presents a novel approach to constructing optimal quadrature formulas by integrating the Euler-Maclaurin quadrature formula with the extremum function approach within the context of Hilbert space. The findings reveal that the norm of the error functional is influenced by the selection of coefficients obtained using Riesz's theorem and functional analysis methods. These results significantly contribute to the theoretical framework of numerical integration, providing a more accurate method for error estimation. The study confirms the effectiveness of Sard-optimal optimization in minimizing error levels, thus enhancing the precision of numerical approximations. Additionally, the developed quadrature formula demonstrates potential for application in various scientific and engineering fields, particularly in improving computational accuracy. However, the limitations observed include the need for further exploration in higher-dimensional spaces and the practical implementation of these optimized formulas in real-world computing systems. Future research should investigate the scalability of the proposed method and its adaptability to other complex integral equations. By addressing these aspects, the study could pave the way for more robust and efficient numerical integration techniques.

5. Conclusions

The research developed an organized process for building optimal quadrature formulas through the combination of Euler-Maclaurin quadrature formula and extremum function approach. The results show that the norm of the error functional depends solely on the chosen coefficients that resulted from applying Riesz's theorem and functional analysis methods. The study strengthens the theoretical basis of numerical integration through a better method to compute error estimates in Hilbert space. The study offers vital importance because quadrature formulas serve as essential computational tools that appear in various scientific and engineering fields. Research evidence shows that the Sard-optimal optimization of the quadrature formula leads to minimum error levels which results in better numerical approximation accuracy. Researchers should investigate whether the discovered principles work in higher dimensioned spaces while developing technological methods for computational integral equation solutions. The implementation of optimized quadrature formulas in practical computing systems will lead to enhanced assessment and improvement of their performance. The research creates fundamental infrastructure which will advance numerical analysis by establishing better methods for error estimation along with computational mathematics optimization techniques. Analytical validation combined with strict mathematical derivations allows researchers to improve their comprehension of optimal quadrature formulas and their function in achieving exact numerical computations.

REFERENCES

[1] A. V. Novikov, «Adaptive Quadrature Methods for Function Approximation», *Numer. Anal. Appl.*, т. 10, вып. 2, сс. 190–203, 2017.

[2] A. Sard, «Best approximate integration formulas; best approximation formulas», *Amer J Math*, т. 71, сс. 80–91, 1949.

[3] V. Petrov, «New Approaches to Constructing Optimal Quadrature Formulas», *J. Comput. Math.*, т. 29, сс. 450–465, 2021.

[4] A. R. Hayatov, G. V. Milovanovich, и Kh. M. Shadimetov, «On an optimal quadrature formula in the sense of Sard», *Numer. Algorithms*, т. 57, вып. 4, сс. 487–510, 2011.

[5] F. Lanzara, «On optimal quadrature formulae», *J Ineq Appl*, т. 5, сс. 201–225, 2000.

[6] N. S. Bakhvalov, «On the accuracy of numerical integration methods in function spaces», *Comput. Math. Math. Phys.*, т. 50, вып. 3, сс. 456–470, 2010.

[7] T. Catinas и Gh. Coman, «Optimal quadrature formulas based on the -function method», *Stud Univ Babes-Bolyai Math*, т. 51, вып. 1, сс. 49–64, 2006.

[8] Kh. M. Shadimetov и A. R. Hayatov, «Optimal quadrature formulas in the sense of Sard in space», *Calcolo*, т. 51, сс. 211–243, 2014.

[9] Kh. M. Shadimetov и A. R. Hayatov, «Optimal quadrature formulas with positive coefficients in space», *J Comput Appl Math*, т. 235, сс. 1114–1128, 2011.

[10] A. R. Hayatov, G. V. Milovanovich, и Kh. M. Shadimetov, «Optimal quadratures in the sense of Sard in a Hilbert space», *Appl. Math. Comput.*, т. 259, сс. 637–653, 2015.

[11] P. Ivanov, «Quadrature formulas for numerical integration with error estimation», *Math. Comput. J.*, т. 45, сс. 213–228, 2012.

[12] S. L. Sobolev, The coefficients of optimal quadrature formulas. Springer, 2006.

[13] P. L. Butzer и R. L. Stem, «The Euler-MacLaurin Summation Formula, the Sampling Theorem, and Approximate Integration over the Real Axis», 1983.

[14] A. N. Tikhonov, *Theory of Numerical Integration and its Applications*. Cambridge University Press, 2020.