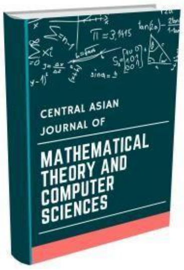

CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES

https://cajmtcs.centralasianstudies.org/index.php/CAJMTCS

Volume: 06 Issue: 2 | April 2025 ISSN: 2660-5309

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Article

The Square of The Norm of The Error Functional Concerning One Quadrature Formula in Sobolev Space

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Abstract: Quadrature formulas are fundamental tools in numerical integration, used to approximate integrals with high accuracy. Their efficiency depends on minimizing the error functional, which is critical in functional analysis and numerical approximation theory. The study focuses on Sobolev spaces, which provide a rigorous framework for analyzing quadrature formulas. In these spaces, error functionals play a crucial role in assessing the accuracy of numerical integration methods. Prior research has explored extremal functions and optimal quadrature formulas, but precise error norm calculations remain an ongoing challenge. Although optimal quadrature formulas have been studied extensively, the explicit calculation of the square of the norm of the error functional in Sobolev spaces requires further exploration. A deeper understanding of extremal functions and their impact on error minimization is necessary to advance numerical integration techniques. This research aims to determine the square of the norm of the error functional for a given quadrature formula in Sobolev spaces, utilizing extremal function analysis and functional optimization methods. The study derives an explicit formula for the norm of the error functional, proving its dependence on the coefficients of the quadrature formula. Using Riesz representation and Green's function techniques, an extremal function corresponding to the error functional is obtained, leading to a rigorous calculation of the error norm. The research presents a precise computation of the error functional norm, contributing to the optimization of quadrature formulas in Sobolev spaces. The findings enhance the theoretical understanding of numerical integration and provide a foundation for developing more accurate computational methods. The results are significant for improving numerical integration techniques used in applied mathematics, physics, and engineering. The optimized quadrature formulas can enhance computational efficiency in solving integral equations, reducing numerical errors in scientific computing applications.

Keywords: Quadrature Formula, Extremal Function, Error Functional, Norm

Citation: Shavkatovich, U. S., ugli, S, B, K., qizi, S, M, K. The Square of The Norm of The Error Functional Concerning One Quadrature Formula in Sobolev Space. Central Asian Journal of Mathematical Theory and Computer Sciences 2025, 6(2), 177-183.

Received: 28th Feb 2025
Revised: 7th Mar 2025
Accepted: 13th Mar 2025
Published: 18th Mar 2025

1. Introduction

The study of quadrature formulas in Sobolev spaces plays a fundamental role in numerical integration, particularly in optimizing computational efficiency and reducing error margins in approximate integral calculations. This research focuses on determining the square of the norm of the error functional for a specific quadrature formula, a critical aspect in evaluating the accuracy and reliability of numerical methods. The Sobolev space framework provides a structured approach for analyzing functions with derivatives in a generalized sense, ensuring a rigorous foundation for error estimation. The study employs extremal function theory and functional analysis techniques to derive the optimal quadrature coefficients that minimize the error functional.

By leveraging the Riesz representation theorem and Green's function methodology, the research calculates the precise error norm, demonstrating its dependence on quadrature coefficients. The findings contribute to the broader field of computational mathematics by offering a systematic approach to refining quadrature formulas, which are widely applicable in solving differential equations, physics simulations, and engineering computations. This study aligns with the foundational work of Sobolev and his successors, further developing optimal numerical integration techniques and expanding their applicability in multidimensional contexts. Understanding these optimal formulas is crucial for advancing numerical methods and ensuring computational accuracy across various scientific and engineering disciplines.

2. Materials and Methods

An important task in the theory of quadrature formulas is to find the maximum error of a quadrature formula for a given class of functions.

Let's consider quadrature formula of the following form

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} k[\beta]\varphi(h\beta), \tag{1}$$

where $k[\beta]$ are the coefficients of the quadrature formula (1), $[\beta]=(h\beta)$, $h=\frac{1}{N}$, $N=1,2,3,...,$ is element of space $W_2^{(m)}(0,1)$ and $k[\beta]=0$ at $h\beta \notin [0,1]$.

The error of the quadrature formula (1) is the difference between the integral and quadrature sum

$$(\ell,\varphi)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)dx=\int_0^1\varphi(x)dx-\sum_{\beta=0}^{N}k[\beta]\varphi(h\beta),$$

where

$$\ell(x)=i_{[0,1]}(x)-\sum_{\beta=0}^{N}k[\beta]\delta(x-h\beta) . \tag{2}$$

Here $\ell(x)$ is the error functional of quadrature formula (1), $i_{[0,1]}(x)$ is indicator of the interval [0,1], $\delta(x)$ is the Dirac delta function.

In this work, the extremal function corresponding to the error functional (2) of the quadrature formula (1) in the space $W_2^{(m)}(0,1)$ is found, and the square of the norm of the error functional is calculated.

The Sobolev space $W_2^{(m)}(0,1)$ is the Hilbert space of classes of real functions $\varphi(x)$ differing at most by a polynomial of degree $m-2$ with derivatives (in the sense of in a generalized) of order m square integrable in the interval (0,1) and an inner product

$$<\varphi,\psi>=\int_0^1\left(\frac{d^m\varphi(x)}{dx^m}\cdot\frac{d^m\psi(x)}{dx^m}+\frac{d^{m-1}\varphi(x)}{dx^{m-1}}\cdot\frac{d^{m-1}\psi(x)}{dx^{m-1}}\right)dx. \tag{3}$$

Then the norm of the function $\varphi(x)$ in the space $W_2^{(m)}(0,1)$ is determined by the formula

$$\|\varphi\|=\left\{\int_0^1\left(\left(\frac{d^m\varphi(x)}{dx^m}\right)^2+\left(\frac{d^{m-1}\varphi(x)}{dx^{m-1}}\right)^2\right)dx\right\}^{\frac{1}{2}}.$$

The error of the quadrature formula is a linear functional in $W_2^{(m)*}(0,1)$, where $W_2^{(m)*}(0,1)$ is conjugate space to the space $W_2^{(m)}(0,1)$, i.e.

$$\ell(x)\in W_2^{(m)*}(0,1).$$

For the error functional $\ell(x)$ belongs to the space $W_2^{(m)*}(0,1)$, it is necessary that

$$\left(\ell\left(x^\alpha\right)\right)=0, \alpha=0,1,...,m-2 . \tag{4}$$

It is natural to evaluate the quality of the quadrature formula (1) using the maximum error of this formula on the unit ball of the Hilbert space $W_2^{(m)}(0,1)$, that is, using the norm of the functional $\ell(x)$:

$$\left\|\ell \middle| W_2^{(m)*}(0,1)\right\| = \sup_{\left\|\varphi \middle| W_2^{(m)}\right\|=1} \left|(\ell,\varphi)\right|.$$

It can be seen that the norm of the error functional $\ell(x)$ depends on the coefficients $k[\beta]$.

If

$$\left\|\mathring{\ell} \middle| W_2^{(m)*}(0,1)\right\| = \inf_{k[\beta]} \left\|\ell \middle| W_2^{(m)*}(0,1)\right\|, \tag{5}$$

then they say that the functional $\mathring{\ell}(x)$ corresponds to the optimal quadrature formula in the space e $W_2^{(m)}(0,1)$.

If you want to find the maximum possible error over the space $W_2^{(m)}(0,1)$ of the constructed quadrature formula, then it is enough to solve the following problem.

Problem 1. Find the norm of the error functional $\ell(x)$ of the quadrature formula (1) in the space $W_2^{(m)}(0,1)$.

If you need to find the optimal quadrature formula by varying the coefficients $k[\beta]$, then you need to solve the following problem.

Problem 2. Find such values of the coefficients $k[\beta]$ that equality (5) is satisfied.

The formulated problems transfer the theory of approximate calculations of integrals to the section of extremal problems of functional analysis, formed in the scientific direction in the 30-50 s of the last century and associated with the name of A.N. Kolmogorov.

In the multidimensional case, the formulation of problems 1 and 2 were set by S.L. Sobolev . Next, he gives an algorithm for constructing optimal lattice cubature formulas in Sobolev space $L_2^{(m)}(R^n)$ [1,2].

Subsequently, Sobolev's research on optimal lattice cubature formulas and asymptotic formulas was and is being developed by his students [3-13].

3. Results and Discussion

Figures In the next paragraph we find the extremal function.

3.1. Extremel function.

In order to solve problem 1, i.e. to find the norm of the error functional (2) in the space $W_2^{(m)*}(0,1)$, the extremal function of this functional is used. Function $U_\ell(x)$ is called an extremal function of the functional $\ell(x)$ [1] if the equality

$$(\ell,U_\ell) = \left\|\ell \middle| W_2^{(m)*}(0,1)\right\| \cdot \left\|U_\ell \middle| W_2^{(m)}(0,1)\right\|. \tag{6}$$

In space $W_2^{(m)}(0,1)$, using the Riesz theorem on the general form of a linear continuous functional on Hilbert spaces, the extremal function is expressed in terms of a given functional and, in addition, the equality is satisfied

$$\left\|\ell \middle| W_2^{(m)*}(0,1)\right\| = \left\|U_\ell \middle| W_2^{(m)}(0,1)\right\|. \tag{7}$$

Therefore, from (6) and (7) we conclude that

$$(\ell,U_\ell) = \left\|\ell \middle| W_2^{(m)*}(0,1)\right\|^2. \tag{8}$$

On the other hand, using the same theorem, for any element $\varphi(x)$ of space $W_2^{(m)}(0,1)$ we obtain

$$(\ell,\varphi) = \langle U_\ell,\varphi\rangle,$$

Where

$$<U_\ell,\varphi>=\int_0^1\left(\frac{d^m\varphi(x)}{dx^m}\cdot\frac{d^mU_\ell(x)}{dx^m}+\frac{d^{m-1}\varphi(x)}{dx^{m-1}}\cdot\frac{d^{m-1}U_\ell(x)}{dx^{m-1}}\right)dx \tag{9}$$

pseudo-inner product in space $W_2^{(m)}(0,1)$. Let $\varphi(x)$ is a function be finite and infinitely differentiable, i.e.

$\varphi(x)\in \overset{\circ}{C}{}^{(\infty)}(0,1)$.

Integrating m the right-hand side of equality (9) by parts, $U_\ell(x)$ we obtain for the functions

$$U_\ell^{(2m)}(x)-U_\ell^{(2m-2)}(x)=(-1)^m\ell(x). \tag{10}$$

It is known [1] that space $\overset{\circ}{C}{}^{(\infty)}(0,1)$ dense in space $W_2^{(m)}(0,1)$. Therefore, we can approximate functions from space as accurately as we like $W_2^{(m)}(0,1)$ using a sequence of functions from $\overset{\circ}{C}{}^{(\infty)}(0,1)$. Then, for anyone $\varphi(x)\in W_2^{(m)}(0,1)$ Integrating by parts the right-hand side of equality (9) we have

$$(\ell,\varphi)=\langle U_\ell,\varphi\rangle=\int_0^1\left(\varphi^{(m)}(x)\cdot U_\ell^{(m)}(x)+\varphi^{(m-1)}(x)\cdot U_\ell^{(m-1)}(x)\right)dx=$$

$$=\sum_{s=1}^{m-1}(-1)^{s+1}\varphi^{(m-2-s)}(x)\left(U_\ell^{(m+s+1)}(x)-U_\ell^{(m+s-1)}(x)\right)\Bigg|_{x=1}^{x=0}+$$

$$+\varphi^{(m-1)}(x)U_\ell^{(m)}(x)\Bigg|_{x=0}^{x=1}+$$

$$+(-1)^m\int_0^1\varphi(x)\left(U_\ell^{(2m)}(x)-U_\ell^{(2m-2)}(x)\right)dx.$$

From here, from arbitrariness $\varphi(x)\in W_2^{(m)}(0,1)$ and from the uniqueness of the function $U_\ell(x)\in W_2^{(m)}(0,1)$ taking into account equality (10) we obtain the following equalities

$$U_\ell^{(2m)}(x)-U_\ell^{(2m-2)}(x)=(-1)^m\ell(x), \tag{11}$$

$$\left(U_\ell^{(m+s+1)}(x)-U_\ell^{(m+s-1)}(x)\right)\Bigg|_{x=0}^{x=1}=0,\ s=\overline{1,m-1}, \tag{12}$$

$$U_\ell^{(m)}(x)\Bigg|_{x=0}^{x=1}=0. \tag{13}$$

Theorem 1. *The solution to the boundary value problem (11)-(13) is an extremal function of the error functional (2) of the quadrature formula (1) and has the form*

$U_\ell(x)=(-1)^m\ell(x)*\mu_m(x)+P_{m-2}(x)$,

where

$$\mu_m(x)=\frac{signx}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{n=1}^{m-1}\frac{x^{2n-1}}{(2n-1)!}\right) \tag{14}$$

fundamental solution of the differential operator $2m-$ *first order, i.e. solutions to the equation*

$$\left(\frac{d^{2m}}{dx^{2m}}-\frac{d^{2m-2}}{dx^{2m-2}}\right)\mu_m(x)=\delta(x),$$

$signx=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0,\end{cases}$ $P_{m-2}(x)$ *is a polynomial of degree* $m-2$, $\delta(x)$ *is the well-known Dirac delta function.*

Proof. It is known that the general solution of an inhomogeneous differential equation is a particular solution of an inhomogeneous differential equation plus a general solution of the corresponding homogeneous differential equation.

As stated above, what $W_2^{(m)}(0,1)$ is a Hilbert space with scalar product (3). Therefore, using the general form of a linear functional in a Hilbert space, we represent the error functional in the form (9), where the $U_\ell(x) \in W_2^{(m)}(0,1)$ Riesz element is called. In addition, according to the Riesz theorem, equality (7) holds. By virtue of (3) and (9), we obtain the following identity, which is valid for any function $\varphi(x) \in W_2^{(m)}(0,1)$

$$\int_0^1 \left(\frac{d^m\varphi(x)}{dx^m} \cdot \frac{d^m U_\ell(x)}{dx^m} + \frac{d^{m-1}\varphi(x)}{dx^{m-1}} \cdot \frac{d^{m-1} U_\ell(x)}{dx^{m-1}} \right) dx = \int \ell(x)\varphi(x)dx \quad . \tag{15}$$

Having performed m the integration by parts on the left side of (15) once, we obtain a boundary value problem in generalized functions

$$\frac{d^{2m}U_\ell(x)}{dx^{2m}} - \frac{d^{2m-2}U_\ell(x)}{dx^{2m-2}} = (-1)^m \ell(x), \tag{16}$$

$$\left(\frac{d^{m+k}U_\ell(x)}{dx^{m+k}} - \frac{d^{m-2+k}U_\ell(x)}{dx^{m-2+k}} \right) \Bigg|_{x=0}^{x=1} = 0, \quad k = \overline{1, m-3}. \tag{17}$$

From the theory of boundary value problems it is known that this problem has a solution that is unique, up to a term, which $P_{m-2}(x)$ is a polynomial of degree $\le m-2$.

All solutions to equation (16) are written in the form

$$U_\ell(x) = (-1)^m \ell(x) * \mu_m(x) + P_{2m-3}(x) + ae^x + be^{-x}, \tag{18}$$

where $P_{2m-3}(x)$ is some polynomial of degree $2m-3$.

In fact, it is easy to check that the function $(-1)^m \ell(x) * \mu_m(x) \in W_2^{(m)}(0,1)$ is a solution to equation (16), and the entire solution to the homogeneous equation

$$\frac{d^{2m}U_\ell(x)}{dx^{2m}} - \frac{d^{2m-2}U_\ell(x)}{dx^{2m-2}} = 0$$

from space $W_2^{(m)}(0,1)$ are polynomials of degree $< 2m-3$, so that the general solution of equation (16) in space $W_2^{(m)}(0,1)$ has the form (1 8).

It is easy to see that in order for the solution $U_\ell(x)$ to satisfy both condition (17), the equality $P_{2m-3}(x) = P_{m-2}(x)$, $a = 0$ and $b = 0$.

Really,

$$\left(\frac{d^{m+k}}{dx^{m+k}} - \frac{d^{m-2-k}}{dx^{m-2-k}} \right) \left(\ell(x) * \mu_m(x) + P_{m-2}(x) \right) = \ell(x) * \mu_m{}^{(m+k)}, \text{ at } k = \overline{0, m-3}.$$

So

$$U_\ell(x) = (-1)^m \ell(x) * \mu_m(x) + P_{m-2}(x) .$$

Theorem 1 has been proven completely.

Now, using the found extremal function, we calculate the square of the norm of the error functional for the quadrature formulas.

3.2. The norm of the error functional for quadrature formulas (1).

Now we can calculate the norm of the error functional. The following theorem is true.

Theorem 2. For the square of the norm of the error functional, the following formula holds:

$$\left\| \ell(x) \middle| W_2^{(m)*}(0,1) \right\|^2 = (-1)^m \left[\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} k[\beta] k[\gamma] \mu_m(h\beta - h\gamma) - \right.$$
$$\left. -2 \sum_{\beta=0}^{N} k[\beta] \int_0^1 \mu_m(x - h\beta) dx + \int_0^1 \int_0^1 \mu_m(x-y) dx dy \right], \tag{19}$$

where $\mu_m(x)$ Green's function is determined by formula (14).

Proof of Theorem 2. Indeed, since the space $W_2^{(m)}(0,1)$ is the Hilbert space, then by Riesz's representation theorem on the general form of a linear functional and taking into account the definition of an extremal function, we have

$$\left(\ell,U_\ell\right)=\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|\cdot\left\|U_\ell\middle|W_2^{(m)}(0,1)\right\|=\left\|U_\ell\middle|W_2^{(m)}(0,1)\right\|^2=\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|^2.$$

Hence, taking into account Theorem 1, we have

$$\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|^2=\left(\ell,U_\ell\right)=\int\ell(x)U_\ell(x)dx=$$

$$=\int\left(i_{[0,1]}(x)-\sum_{\beta=0}^{N}k[\beta]\delta(x-h\beta)\right)\times$$

$$\times\left((-1)^m\ell(x)*\frac{signx}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{n=1}^{m-1}\frac{x^{2n-1}}{(2n-1)!}\right)+P_{m-2}(x)\right)dx.$$

Using equality (4) we obtain

$$\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|^2=\int\ell(x)\left((-1)^m\ell(x)*\frac{signx}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{n=1}^{m-1}\frac{x^{2n-1}}{(2n-1)!}\right)\right)dx. \quad (20)$$

First, we calculate the following convolution

$$\ell(x)*\frac{signx}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{n=1}^{m-1}\frac{x^{2n-1}}{(2n-1)!}\right)$$

$$\ell(x)*\frac{signx}{2}\left(\frac{e^x-e^{-x}}{2}-\sum_{n=1}^{m-1}\frac{x^{2n-1}}{(2n-1)!}\right)=$$

$$=\int_{-\infty}^{\infty}\ell(y)*\frac{sign(x-y)}{2}\left(\frac{e^{x-y}-e^{y-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-y)^{2n-1}}{(2n-1)!}\right)dy=$$

$$=\int_{-\infty}^{\infty}\left(i_{[0,1]}(y)-\sum_{\beta=0}^{N}k[\beta]\delta(y-h\beta)\right)*\frac{sign(x-y)}{2}\left(\frac{e^{x-y}-e^{y-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-y)^{2n-1}}{(2n-1)!}\right)dy=$$

$$=\int_0^1\frac{sign(x-y)}{2}\left(\frac{e^{x-y}-e^{y-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-y)^{2n-1}}{(2n-1)!}\right)dy-$$

$$-\sum_{\beta=0}^{N}k[\beta]\frac{sign(x-h\beta)}{2}\left(\frac{e^{x-h\beta}-e^{h\beta-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-h\beta)^{2n-1}}{(2n-1)!}\right). \quad (21)$$

Then substituting (21) into (20), we get

$$\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|^2=(-1)^m\int\left(i_{[0,1]}(x)-\sum_{\beta=0}^{N}k[\beta]\delta(x-h\beta)\right)\times$$

$$\times\left(\int_0^1\frac{sign(x-y)}{2}\left(\frac{e^{x-y}-e^{y-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-y)^{2n-1}}{(2n-1)!}\right)dy-\right.$$

$$\left.-\sum_{\gamma=0}^{N}k[\gamma]\frac{sign(x-h\gamma)}{2}\left(\frac{e^{x-h\gamma}-e^{h\gamma-x}}{2}-\sum_{n=1}^{m-1}\frac{(x-h\gamma)^{2n-1}}{(2n-1)!}\right)\right)dx.$$

From here, opening the brackets and simplifying, we get

$$\left\|\ell(x)\middle|W_2^{(m)*}(0,1)\right\|^2=(-1)^m\left[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}k[\beta]k[\gamma]\mu_m(h\beta-h\gamma)-2\sum_{\beta=0}^{N}k[\beta]\int_0^1\mu_m(x-h\beta)dx+\right.$$

$$\left.+\int_0^1\int_0^1\mu_m(x-y)dxdy\right]. \quad (22)$$

From equality (22), we immediately obtain (19). Theorem 2 is completely proved.

5. Conclusions

In this work, in the factorized space $W_2^{(m)}(0,1)$ an extremal function corresponding to the error functional of quadrature formulas is found. Moreover, in the conjugate space $W_2^{(m)*}(0,1)$ the square of the norm of the error functional has been calculated.

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