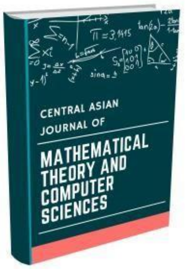

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Article

Existence and Uniqueness of The Solution of The Exact Problem For The Integro-Differential Heat Distribution Equation

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Abstract: The study of integro-differential equations plays a fundamental role in mathematical physics, particularly in the analysis of heat dissipation processes. The existence and uniqueness of solutions to such equations are crucial for ensuring the reliability of theoretical models. The Cauchy problem for the integro-differential heat dissipation equation can be reformulated into an equivalent Volterra integral equation. Traditional approaches employ fundamental solutions and functional series to establish solvability conditions. While various studies have explored heat conduction problems with memory effects, there remains a need for rigorous proofs ensuring the uniqueness of solutions in the space of Hölder functions. This study aims to establish a complete proof of the existence and uniqueness of the solution to the integro-differential heat dissipation equation by utilizing the method of successive approximations and integral inequalities. The research demonstrates that the functional series converges uniformly within the given domain, ensuring the existence of a solution. Furthermore, through the application of the Gronwall–Bellman inequality, it is shown that the solution is unique. The use of the Hölder function space in proving the uniqueness and existence of solutions offers a refined approach to analyzing heat dissipation equations, strengthening the theoretical foundations of inverse problem theory. The findings contribute to mathematical physics by providing a rigorous framework for modeling heat distribution processes and ensuring the stability of integro-differential equation-based models in applied sciences.

Keywords: Integro-differential, Heat Dissipation Equation, Direct Problem, Theorem, Lemma

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1. Introduction

In the study of integro-differential equations, the Cauchy problem plays a crucial role in understanding complex physical and mathematical phenomena. These problems, particularly in the context of heat dissipation and inverse problem theory, provide fundamental insights into mathematical modeling and real-world applications.

The focus of this study is on solving a specific Cauchy problem involving an integro-differential equation with an integral term of the Volterra type. By transforming this problem into an equivalent integral equation, we aim to analyze the existence and uniqueness of the solution within the given function space.

This paper explores the methodological framework necessary to establish the well-posedness of the problem, leveraging integral transformations and approximation techniques. Through this approach, we investigate how the problem can be reformulated and solved efficiently while ensuring its stability and convergence.

$(x,y,t) \in \mathbb{R}_T^2\{(x,y,t)|x,y \in \mathbb{R}^2, 0 \le t < T\}$ we consider the problem of determining the function $u(x,y,t)$, in the following field:

$$u_t - \Delta u + h(x)u(x,y,t) = \int_0^t k\,(x,\tau)u(x,y,t-\tau)d\tau, (x,y,t) \in \mathbb{R}_T^2, \qquad (1)$$

$$u|_{t=0} = \varphi(x,y), \ (x,y) \in \mathbb{R}^2, \qquad (2)$$

here $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ - Laplace operator.

The problem of finding the function $u(x,y,t)$ from the integro-differential equation (1) for a given function k using the initial condition (2) is called the Cauchy problem [1]. Such problems are called proper problems in the theory of inverse problems. The solution of the Cauchy problem (1) and (2) is equivalent to an integral equation of the Volterra type. For this, we use the following formula:

$$z(x,y,t) = \int_{\mathbb{R}^2} \varphi\,(\xi)G(x-\xi;t)d\xi + \int_0^t d\tau \int_{\mathbb{R}^2} F\,(\xi,\tau)G(x-\xi;t-\tau)d\xi \quad (3)$$

Formula (3) represents the solution to the Cauchy problem for the following variable coefficient heat dissipation equation:

$$z_t - \Delta z = F(x,y,t), \qquad (x,y) \in \mathbb{R}^2, t > 0,$$
$$z(x,y,0) = \varphi(x,y), \qquad x \in \mathbb{R}^2.$$

$G(x-\xi_1, y-\xi_2; t-\tau) = \frac{1}{4\pi(t-\tau)} e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4(t-\tau)}}$ function $\frac{\partial}{\partial t} - \Delta$ is a fundamental solution of the differential operator with constant coefficients, where $\xi = (\xi_1, \xi_2)$, $d\xi = d\xi_1 d\xi_2$. (3) Using formula (1), (2), we express the Cauchy problem in the form of the following Volterra integral equation of the second kind:

$$u(x,y,t) = \int_{\mathbb{R}^2} \varphi\,(\xi)G(x-\xi_1, y-\xi_2; t)d\xi_1 d\xi_2 +$$
$$+\int_0^t d\tau \int_{\mathbb{R}^2} \left[\int_0^\tau k\,(\xi',\alpha)u(\xi,\tau-\alpha)d\alpha - h(\xi)u(\xi,\tau)\right] \times$$
$$\times G(x-\xi_1, y-\xi_2; \tau)d\xi_1 d\xi_2. \qquad (4)$$

From $\overline{\mathbb{R}}_T^2 = \{(x,y,t)|x,y \in \mathbb{R}^2, 0 \le t \le T\}$, $\overline{\mathbb{R}}_T, = \{(x',t)|x' \in \mathbb{R}, 0 \le t \le T\}$, we can know.

2. Materials and Methods

Lemma. Imagine, $\varphi(x,y) \in H^{l+2}(\mathbb{R}^2), h(x,y) \in H^l\left(\overline{\mathbb{R}}_T^2\right)$, $k(y,t) \in H^{l,l/2}(\overline{\mathbb{R}}_T)$ *and* $p_0\mathrm{T} + p_0\frac{T^2}{2!} < 1$. Then the integral equation (4) has a unique solution $u(x,y,t)$ belonging to the class $H^{l+2,(l+2)/2}(\overline{\mathbb{R}}_T^2)$, where $l \in (0,1)$.

Proof. In proving Maskur's lemma, we use the method of successive approximation [2]. To implement the principle of successive approximation for the integral equation (4), we construct the following sequence:

$$u_0(x,y,t) = \int_{\mathbb{R}^2} \varphi\,(\xi)G(x-\xi_1, y-\xi_2; t)d\xi_1 d\xi_2$$

$$u_1(x,t) = \int_0^t d\tau \int_{\mathbb{R}^2} \left[\int_0^\tau k\,(\xi_1,\alpha)u_0(\xi_1,\xi_1,\tau-\alpha)d\alpha - h(\xi_1,\xi_1)u_0(\xi,\tau)\right] \times$$
$$\times G(x-\xi_1, y-\xi_2; t-\tau)d\xi_1 d\xi_2,$$

$$u_1(x,t) = \int_0^t d\tau \int_{\mathbb{R}^2} \int_0^\tau k\,(\xi_1,\alpha)u_0(\xi_1,\xi_1,\tau-\alpha)G(x-\xi_1, y-\xi_2; t-\tau)\,d\xi$$

$$u_2(x,t)=\int_0^t d\tau \int_{\mathbb{R}^2}\left[\int_0^\tau k\,(\xi_1,\alpha)u_1(\xi_1,\xi_1,\tau-\alpha)d\alpha-h(\xi_1,\xi_1)u_1(\xi,\tau)\right]\times$$
$$\times\, G(x-\xi_1,y-\xi_2;t-\tau)d\xi_1 d\xi_2,$$

……………….

$$u_m(x,t)=\int_0^t d\tau \int_{\mathbb{R}^2}\left[\int_0^\tau k\,(\xi_1,\alpha)u_{m-1}(\xi_1,\xi_1,\tau-\alpha)d\alpha-h(\xi_1,\xi_1)u_{m-1}(\xi_1,\xi_1,\tau)\right]$$
$$\times\, G(x-\xi_1,y-\xi_2;t-\tau)d\xi_1 d\xi_2, \qquad (5)$$

……………….

Using $\varphi_0=|\varphi(x,y)|^l$, $h_0=|h(x,y)|^l$, $p_0=\max\{h_0,\ k_0\}$, on $\mathbb{R}^2_T$ (5) we evaluate the functions $u_m(x,y,t)$ defined using the integral in a modular way:

$$|u_0(x,y,t)|_{H^{l+2,\frac{l+2}{2}}}\le\varphi_0$$

the same [3]

$$|u_1(x,y,t)|_{H^{l+2,\frac{l+2}{2}}}\le\varphi_0 p_0\left(\mathrm{t}+\frac{t^2}{2!}\right),$$

here $k_0:=|k(x,t)|_T^{l,l/2}$. We also perform the evaluation for $u_2(x,y,t)$:

$$|u_2(x,y,t)|_{H^{l+2,\frac{l+2}{2}}}\le\varphi_0 p_0^2\left(\mathrm{t}+\frac{t^2}{2!}\right)^2,$$

……………….

$$|u_m(x,y,t)|_{H^{l+2,\frac{l+2}{2}}}\le\varphi_0 p_0^m\left(\mathrm{t}+\frac{t^2}{2!}\right)^m,$$

…………..

$$\int_{\mathbb{R}^n} G\,(x-\xi_1,y-\xi_2;\theta(t))d\xi=1. \qquad (6)$$

We can construct the following functional array:

$$\sum_{j=0}^{\infty}u_j\,(x,y,t).$$

Using the above estimates, we equate the resulting functional series with a numerical series in the domain $(x,y,t)\in\mathbb{R}^2_T$ as follows:

$$\sum_{m=0}^{\infty}|\,u_j(x,y,t)|\le\sum_{m=0}^{\infty}\varphi_0 p_0^m\left(\mathrm{T}+\frac{T^2}{2!}\right)^m.$$

By the condition of the theorem, the finite series converges [4]. The functional series is convergent according to the Weierstrass sign of the smooth convergence of functional series [5]. From this result, the sequence of functions $u_j(x,y,t)$ defined using the integral equation (4) smoothly converges to some function $u(x,y,t)$ defined in the function space $H^{l+2,(l+2)/2}(\mathbb{R}^2_T)$ [6]. Thus, we have shown that (1)-(2) Cauchy problem has a solution belonging to the class $H^{l+2,(l+2)/2}(\mathbb{R}^2_T)$ [7].

3. Results and Discussion

We have seen that the integral equation (4) has a solution [8]. Now we will show that this solution is unique [9]. For this, let us assume the opposite, that is, let the integral equation (4) have two exactly unequal solutions $u^1(x,y,t)$ and $u^2(x,y,t)$:

$$u^1(x,y,t)=\int_{\mathbb{R}^2}\varphi\,(\xi_1,\xi_1)G(x-\xi_1,y-\xi_2;t)d\xi_1 d\xi_2+$$
$$+\int_0^t d\tau\int_{\mathbb{R}^2}\left[\int_0^\tau k\,(\xi_1,\alpha)u^1(\xi_1,\xi_1,\tau-\alpha)d\alpha-h(\xi_1,\xi_1)u^1(\xi_1,\xi_1,\tau)\right]\times$$

$$\times G(x-\xi_1, y-\xi_2; t-\tau)d\xi_1 d\xi_2$$

and

$$u^2(x,y,t) = \int_{\mathbb{R}^2} \varphi\,(\xi_1,\xi_1)G(x-\xi_1,y-\xi_2;t)d\xi_1 d\xi_2 +$$

$$+\int_0^t d\tau \int_{\mathbb{R}^2} \left[\int_0^\tau k\,(\xi_1,\alpha)u^2(\xi_1,\xi_1,\tau-\alpha)d\alpha - h(\xi_1,\xi_1)u^2(\xi_1,\xi_1,\tau)\right]$$

$$\times G(x-\xi_1, y-\xi_2; t-\tau)d\xi_1 d\xi_2.$$

Looking at the difference between the functions u^1 and u^2 we denote their difference by

$$\omega(x,y,t) = u^1(x,y,t) - u^2(x,y,t)$$

As a result

$$\omega(x,y,t) = \int_0^t d\tau \int_{\mathbb{R}^2} \left[\int_0^\tau k\,(\xi_1,\alpha)\omega(\xi_1,\xi_1,\tau-\alpha)d\alpha - h(\xi_1,\xi_1)\omega(\xi_1,\xi_1,\tau)\right]$$

$$\times G(x-\xi_1, y-\xi_2; t-\tau)d\xi_1 d\xi_2. \qquad (7)$$

We obtain a homogeneous Volterra integral equation of the second kind [10]. For each assigned $t \in [0,T]$, we define the modular supremum of the function $\omega(x,t)$ with respect to $x \in \mathbb{R}^n$ by $\widetilde{\omega}(t)$, that is:

$$\widetilde{\omega}(t) = \sup_{x\in\mathbb{R}^2} |\,\omega(x,y,t)|, \qquad t \in [0,T].$$

Then from the integral equation (7)

$$\widetilde{\omega}(t) \le k_0 T \int_0^t \widetilde{\omega}\,(\tau)d\tau, \qquad t \in [0,T]$$

integral inequality is obtained [11]. According to the Growell–Bellman inequality, the last integral inequality has only one solution $\widetilde{\omega}(t) \equiv 0 \;\; t \in [0,T]$ [12]. From this it follows that in the domain $\overline{\mathbb{R}}_T^2$ $\omega(x,y,t) \equiv 0$ or $u^1(x,y,t) = u^2(x,y,t)$ [13]. Thus, the integral equation (4) has a [14] unique solution. The lemma is proven [15].

4. Conclusion

The space of Hölder functions used in the proof of the main result of the existence and uniqueness of a proper solution to the two-dimensional integro-differential heat dissipation equation is presented, as well as an estimate of the Cauchy problem for the heat dissipation equation in the space of Hölder functions.

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