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Article

A New Method for Solving The Fractional Spectral Collocation Equation

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Abstract: This research presents a fractional discrete collision spectral method (FSCM) for solving fractional-order partial differential equations (FPDEs) such as the Borgers equation and the Fokker-Planck equation. The method is based on constructing exact fractional numerical derivatives using a fractional Lagrange interpolator that satisfies the Kronecker delta property at collision points. Fractional PDEs are developed based on several proposed points, including the roots of fractional Jacobian polynomials, achieving exponential convergence in solutions. The study includes an analysis of numerical matrices of fractional derivatives and a comparison of applications in multiple time-invariant and constant-time problems. The method is characterized by ease of implementation and reduced computational cost compared to traditional methods.

Keywords: Fractional Partial Differential Equations, Collision Spectral Method, Fractional Lagrange Interpolators, Exponential Convergence, Borgers Equation And Fokker-Planck Equation

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1. Introduction

The concept ofifractionalicalculusiand fractional-order differential operators are used in the modeling of various physical systems, including viscous fluid flows caused byithe cumulativeimemory effectjofiwall-friction media (whether porousi or fractured), bioengineering applications, and viscoelastic materials. Additionally, anomalous diffusion with nonexponential relaxation patterns governs the transport dynamics in these complex systems. It is discovered that a time-fractional diffusion equation governs the evolution ofithe probability density function of these non-Markovian systems. Formulas for fractional partial differential equations (FPDEs) such the Burgers, Fokker-Planck, advection-diffusion, and fractional-order multitier equations have been quickly generalized from the idea of fractional derivatives[1].

It is not an easy task to convert the numerical methods and references used for differential equations of integer order to fractional differential equations [2] .Due to their dependency on long-range histories, approximating fractional-order systems is computationally intensive, which is the fundamental issue in simulation[3]. While numerical techniques in this field have evolved rapidly in recent years, their history is brief. The Legendre spectral collocation method for solving fractional differential equations is based on expanding the solution using a Legendre manifold and applying a collocation method using relevant nodes such as Gauss-Legendre points[4].

Spectral collocation is a numerical method that approximates the solution by expanding the solution function with basis functions (polynomials such as Chebyshev or

Legendre polynomials) [5] ,Specific polynomials (collocation points) that are more demanding than simple points or Lobatto polynomials[6].

Fractional Chebyshev collocation is a numerical method for solving fractional differential equations where the solution is approximated using Chebyshev polynomials and the equations are applied at Chebyshev–Gauss–Lobatto points[7].

Fractional Lagrange Interpellants

To avoid assessing the inner products in spectral techniques of the Galleria and PG types, typical collocation methods rely on interpolation operators. Finally, {xi}N is defined as aicollection of interpolation points for this purpose [2].

I=1, from which the related Lagrange interpellants are derived. Also, in'order to create a collocation technique, the residualimust disappear on theisame grid points, which are referred to as collocation points. $\{y_i\}N$

I=1. The positions where the residuals vanish need not coincide with the spots where the interpolation takes place. An innovative spectral theory has provided us with the fractional by which we resolve FSLPs in collocation schemes[8].

${}_0D_t^{\check{\iota}}$ μ(x ,t)=£^{v}μ$(x\ ,t)$, $ix \in [\,-1\ ,1]$, $t \in [{}'0\ ,T]$,v … (1)

μi(x ,0) = g (x),

μ (-1' , t) =0 , 'v ∈ (0,1),

μ (-1' ,t) =μ ('1 ,t) =0 , 'v∈ (1 ,2),

The fractional differential operator is denoted by£^{v} , and ĭ is an element of the interval∈ (0 ,1) . "Where v" means the highest fractional order. In terms of novel fractal basis functions that do not contain polynomials, we express the answer to (1). These are known as Jacobi polyfractonomials. For first-kind FSLP, these are the Eigen functions that can be formally derived as

$P_n^{\alpha,\beta,\mu}$('x) =$(1+x)^{-\beta+\mu-1}$ $P_{n-1}^{\alpha-\mu+1\ ,-\beta+\mu-1}$ (x) , ix∈ $[-1,1]$, ...(2)

In the case when$-1 \le \alpha < 2-\mu\ ,and\ -1 \le \beta < \mu-1$ the standard Jacobi polynomials are denoted as $P_{n-1}^{\alpha-\mu+1\ ,\beta+\mu-1}$. When used as basic functions, t Eigen functions with α=β have the same approximation feature. So, the polyfractonononmial Eigen functions that correspond to $\alpha = \beta = -1$ are considered as

Where ('x) are theistandardiJacobiiPolynomialsiin which iμ∈ (0,1),

${}^{(1)}iP_n^{\mu}$(x) =$(1+x)^{\mu}$ $P_{n-1}^{-\mu,\mu}$(x) , i x∈ $[-1,1]$ …(3)

In both that Riemann-Lowville and theiCaputo meanings, theileft-sidedjfractional derivativeiof (3) is provided by the Eigen solution characteristics in [9].

${}_{-1}D_x^{\mu}$ (${}^{(1)}P_n^{\mu}$(x))=$\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}(x)$, …(4)

$P_{n-1}('x)$represents a Legendreipolynomialjof order ('n-1). In ourifractional icollocation approach, such pursueisolutions[10].

$u_N \in V_N^{\mu} = span\{{}^{(1)}P_n^{\mu}(x), 1 \le n \le N\}$, … (5)

1μ ∈ (0,1), $ix \in [-1,1]$, of the from

$u_N(x) = \sum_{j=1}^{N} \breve{u}_j\, {}^{(1)}P_J^{\mu}(x)$ . ... (6)

Another way to describe this polyfractonomial modal expansion is as a nodal expansion, which looks like

$u_N(x) = \sum_{j=1}^{N} u_N\ \ (x_j) h_j^{\mu}(x)$, …(7)

The following four interpolation points are used to create the fractional Lagrange interpolants: $h_j^{\mu}(x)$, where x ranges from $-1 = x_1 < x_2 < \cdots < x_N = 1.$. They are defined as interpolants $h_j^{\mu}(x)$) of fractional order (N+μ-1).

$$h_j^{\mu}(x) = \left(\frac{x-x_1}{x_j-x_1}\right)^{\mu} \prod_{\substack{k=1\\k\neq j}}^{N} \left(\frac{x-x_k}{x_j-x_k}\right)\ ,2 \le j \le N \quad \dots (8)$$

Prior to solving (1), we need to set the superscript μ interpolation parameter. It should be noted that a generic FPDE can, in fact, be linked to many fractional differentiation ordersv$_k$, where $k = 1,2,3, \dots K$ and K is a positive integer. The fractional orders v$_k$, which are provided in the problem, will be used to determine how to set μ..[11].

Notice 1. Due toithe homogeneous iDirichlet boundaryicondition (8)'in (1), weican only build the fractional Lagrange interpolants$h_j^{\mu}(x)$ for j=2,3…$h_j^{\mu}(x), j = 12,3 \dots N-1$ whenithe highest fractionaliorder $iv \in (0,1)$, andiwe impose $iu_N(\pm 1) = 0$ .. At the interpolation point, the fractional interpolants displayed in equation (8) are equal to $h_j^{\mu}(x_k) = \delta_{jk}$, which is the Kroenke delta property; nonetheless, their values fluctuate as an apolyfractonomial function of x_k .The interpolants are used asifractional nodalibasis functions in equation (7), where they imitate the fundamental structure of the Eigen functions (3), which areiused as fractionalimodalibases in the expansion.

FractionaliDifferentiationiMatrix D_σ ,'$0 < \sigma < 1$

We such that differentiationimatrix D_σ of aigeneral fractionaliorder $\sigma \in ('0,1)$ .Weisubstitute (8) in'(7) and take away the σ the orderifractionaliderivative as

$${}_{-1}D_x^{\sigma}\, u_n\,(x) = {}_{-1}D_X^{\sigma}\left[\sum_{j=2}^{N} u_N(x_j)\, h_j^{\mu}(x)\right] \qquad \dots(9)$$

$$= \sum_{j=2}^{N} u_N(x_j)\ {}_{-1}D_x^{\sigma}\left[h_j^{\mu}(x)\right]$$

$$= \sum_{j=2}^{N} u_N(x_j)\,{}_{-1}D_X^{\sigma}\left[\left(\frac{x-x_1}{x_j-x_1}\right)^{\mu} \prod_{\substack{k=1\\k\neq j}}^{N}\left(\frac{x-x_k}{x_j-x_k}\right)\right]$$

$$= \sum_{j=2}^{N} u_N(x_j)\ {}_{-1}D_X^{\sigma}\left[(1+x)^{\mu}\mathcal{G}_j\right]a_j$$

Where $a_j = \frac{1}{(x_j-x_1)^{\mu}}$,and $\mathcal{G}_j = \prod_{\substack{k=1\\k\neq j}}^{N}\left(\frac{x-x_k}{x_j-x_k}\right)$,$j = '2,3,\dots.,N$, are allipolynomials ofiorder $('N-1)$,that can be represented iexactly in terms ofiJacobi polynomialsi $P_{n-1}^{-\mu,\mu}(x)$ as

$$\mathcal{G}_j = \sum_{n=1}^{N} \beta_n^j P_{n-1}^{-\mu,\mu}(x)\ \dots\dots \qquad \dots(10)$$

$${}_{-1}D_x^{\mu}u_N(x) = \sum_{j=2}^{N} u_N(x_j)\ {}_{-1}D_x^{\sigma}\left[(1+x)^{\mu}\sum_{n=1}^{N}\beta_n^j\, P_{n-1}^{-\mu,\mu}(x)\right]a_j \qquad \dots(11)$$

$$= \sum_{j=2}^{N} u_N(x_j)a_j \sum_{n=1}^{N}\beta_{n-1}^j{}_{-1}D_x^{\sigma}\left[(1+x)^{\mu}P_{N-1}^{-\mu,\mu}(X)\right] \text{ by (3)}$$

$$= \sum_{j=2}^{N} u_N(x_j)a_j\ \sum_{n=1}^{N}\beta_n^j\ {}_{-1}D_x^{\sigma}\left[{}^{(1)}P_n^{\mu}(x)\right] \text{ by (4).}$$

(I)The particularicase $\sigma = i\mu \in (0,1)$ in this case theiproperty (4) and obtain that

$${}_{-1}D_X^{\sigma}u_N(x) = \sum_{j=2}^{N} u_N\ (x_j)a_j\sum_{n=1}^{N}\beta_n^j\left[\frac{\Gamma(n+\mu)}{\Gamma(n)}\right]P_{n-1}(x) \qquad \dots(12)$$

Consequently ,we take the interpolation and collocation points to be identical ,also recalling Remark 1 and by evaluating ${}_{-1}D_x^{\mu}u_N(x)$ at the collocation points $\{x_i\}_{i=2}^{N}$ we obtain

$${}_{-1}D_x^{\mu}u_{N\ \ (x)}\Big|_{x_i} = \sum_{j=2}^{N} u_N(x_j)\, a_j\ \sum_{n=1}^{N}\beta_n^j\left[\frac{\Gamma(n+\mu)}{\Gamma(n)}\ P_{n-1}(x_i)\right] \qquad \dots(13)$$

$$= \sum_{j=1}^{N} D_{j=2}^{\mu}\ u_N(x_j)$$

Where D_{ij}^{μ} are the entries of the$(N-1)\times(N-1)$ fractional differentiation matrix D^{μ} obtained as

$$D_{ij}^{\mu} = '\frac{1}{(x_j+1)^{\mu}}\sum_{n=1}^{N}\frac{\Gamma(n+\mu)}{\Gamma(n)}\beta_n^j P_{n-1}(x_j)\ \dots\dots\dots \quad (14)$$

(II)The generalicase $\sigma \in ('0,1)$ .The caseiis importants when that fractional differentialioperator is associatediwith multiple fractionaliderivatives ofidifferent order .

To obtain that fractional differentiation matrixiin this case ,weiperform an fine mapping from $ix \in [-1,1]$ to$s \in [0,1]$ through $is = \frac{(x+1)}{2}$ and rewrite (11) such as

$${}_{-1}D_x^{\sigma} u_N(x) = \sum_{j=2}^{N} u_N(x_j) a_j \sum_{n=1}^{N} \beta_n^j \, {}_{-1}D_{x(s)}^{\sigma} \left[{}^{(1)}P_n^{\mu}(x(s))\right] \ldots \; (15)$$

$$= \sum_{j=2}^{N} u_N(x_j) a_j \sum_{n=1}^{N} \beta_n^j \left(\frac{1}{2}\right)^{\sigma} \, {}_0D_S^{\sigma}\left[{}^{(1)}p_n^{\mu}(x(s))\right],$$

Where ${}^{(1)}P_n^{\mu}(x(s))$ denotes the shifted basis that can be represented as

$${}^{(1)}P_n^{\mu}(x(s)) = 2^{\mu} \sum_{q=0}^{n-1} -1^{n+q-1\prime} \binom{n-1+q}{q}' \binom{n-1+\mu}{n-1-q} s^{q+\mu} \ldots (16)$$

Substituting (16) into (15) we have ${}_{-1}D_{j=2}^{\sigma}(u_N)(x) =$

$$2^{\mu-\sigma} \sum_{j=2}^{N} u_N(x_j) a_j \sum_{n=1}^{N} \beta_n^j \sum_{q=0}^{n-1} -1^{n+q-1} \binom{n-1+q}{q} \binom{n-1+\mu}{n-1-q} \, {}_0D_S^{\sigma}[s^{q+\mu}]$$

In which ${}_0D_S^{\sigma}[s^{q+\mu}]$ can that evaluatediexactly by (8) ,andifinally by inverse transformation we obtain thatiσ-fractional derivativeiof theiapproximateisolution such as

$${}_{-1}D_x^{\sigma} u_N(x) = \sum_{j=2}^{N} u_N(x_j)\left[a_j \sum_{n=1}^{N} \beta_n^j \sum_{q=[\sigma-\mu]}^{n-1} b_{nq}('x+1)^{q+\mu-\sigma}\right] \,..\, 17$$

Iniwhich $[\sigma - \mu]$ denotes that ceilingiof σ-μ and

$$b_{nq} = i{-}1^{n+q-1} \left(\frac{1}{2}\right)^q \quad \binom{n-1+q}{q} \binom{,n-1+\mu}{n-1-q} \frac{\Gamma(q+\mu+1)}{\Gamma(q+\mu-\sigma+1)} \ldots (18)$$

Now that ,similarly byievaluating ${}_{-1}D_x^{\mu} u_N(x)$ at theicollocation point $\{x_i\}_{I=2}^{N}$,

$${}_{-1}D_x^{\sigma} u_N(x) \Big|_{x_i} = \sum_{j=2}^{N} u_N(x_j) \left[a_j \sum_{n=1}^{N} \beta_n^j \sum_{q=[\sigma-\mu]}^{n-1} b_{nq}('x_i+1)^{q+\mu-\sigma}\right]$$

$$= \sum_{j=2}^{N} D_{ij}^{\sigma} u_n(x_j)$$

Example 2: Caputo fractional differential equation:

C D^{\alpha} u(x) = f(x), \quad x \in (0,1), \quad 0 < \alpha < 1^{} [4].

With an initial condition:

u(0) = 0

Methods:

a. Use the representation of N-th degree Chebyshev polynomials.
b. Use Caputo operation matrix for fractional derivative calculations.
c. Choose Chebyshev-Gauss-Lobatto points as collocation points.
d. Form an algebraic system to solve for the coefficients of u_j [7].

Example 1: Nonlinear fractional differential equation:

C D^{\alpha} u(x) = u^2(x) + x^2, \quad x \in (0,1), \quad u(0) = 0, \quad 0 < \alpha < 1^{} \quad 0 < \alpha < 1

Methods:

a. Use Chebyshev spectral rectification.
b. Convert fractional derivatives into operation matrices.
c. Use Newton's method (or Newton-Raphson method) to solve nonlinear systems.
d. Better accuracy than finite differential methods [5].

2. Conclusion

For the purpose of solving both that steady-state and the time-dependentiFPDEs[12], we created an exponentially accurate FSCM in this research.Our next step was to introduce fractionaliLagrange interpolants that meet the Kroenke deltaiproperty at certainicollocation locations. We built it according to a spectral theory for FSLPs that was developed [3]. In order to examine the fractional collocation method's numerical performance, we also solved many linear and nonlinear FPDEs and produced the related fractional differentiation matrices. In order to do this, we presented other potential options for theicollocation interpolation points, namely, the roots of the Jacobiipolyfractonomial ${}^{(1)}P_M^{\mu}(x)$ and that roots of ${}_{-1}D_x^{\mu}\left[{}^{(1)}P_M^{\mu}\right]$ which are the fractional extrema of theipolyfractonomial.iWe contrastedithese novel subsets ofiresidual-vanishing points with the previously established conventional collocation sites for interpolation, including equidistant points, roots of Chebyshevipolynomials, and extrema of Chebyshevipolynomials [4]. We quantitatively proved that, out of all the solutions that lead to the minimum conditioninumber in the corresponding linearisystem and the quickest decline of theiL^∞-norm error, the roots of ${}_{-1}D_x^{\mu}\left[{}^{(x)}P_M^{\mu}(x)\right]$ are the best. Time and the space fractionaliadvectionidiffusion equation [13]. space fractionalimultitier FPDEs, space fractional Burgersiequation, and other steady-state issues were taken into account. We found numerical evidence that the fractional collocation approach converges exponentially [14]. We go over the results of FSCM and how they stack up against other methods. Among existing finite difference schemes and high-order Galerkin spectral techniques for first-order partial differential equations (FPDEs), our FSCM scheme has several benefits, such as (i) ease ofiimplementation, (ii) reduced computingicost, and (iii) exponentialiaccuracy. Our joint work on these aspects is detailed below[15].

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