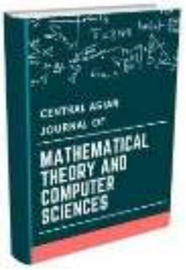

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Article

The Properties Of z-Essential (z-Closed) Fuzzy Submodules

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Abstract: The theory of fuzzy submodules, rooted in the broader domain of fuzzy set theory, provides a nuanced framework for studying algebraic structures under uncertainty. Traditional studies on essential and closed fuzzy submodules have offered important foundations, yet they fall short in addressing deeper generalizations required for more complex fuzzy module interactions. While essential and closed fuzzy submodules are well-established, their extensions namely, -essential and -closed fuzzy submodules have received limited attention, particularly concerning their characterizations and behavior under module homomorphisms. This study aims to define and investigate the properties of -essential and -closed fuzzy submodules within -fuzzy modules, exploring their structural implications and interrelations with classical fuzzy submodule concepts. The research establishes that -essential fuzzy submodules generalize essential ones and introduces conditions under which a -essential submodule becomes essential. Similarly, it proves that every -closed submodule is closed, though not conversely, and provides conditions for closed submodules to be -closed. Moreover, the behavior of -closed submodules under homomorphisms and their preservation through module operations is rigorously examined. This study pioneers the formalization of -essential and -closed fuzzy submodules, offering novel characterizations and multiple propositions that extend classical fuzzy module theory. These results enhance the understanding of structural properties in fuzzy algebra and provide a basis for further exploration in module theory under fuzzified environments, with potential applications in computational algebra and systems modeling where partial membership and graded structures are relevant.

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1. Introduction

Rabi fuzzified concepts to the essential fz-submodules of a fz-module χ,where a fz-submodule κ is called essential (briefly $\kappa \leq_e \chi$), if $\kappa \cap \rho \neq 0_1$, for any non-trivial fz-submodule ρ of χ and a proper fz-submodule κ is called prime fz-submodule whenever $r_t a_\iota \subseteq \kappa$ for fz-singleton r_t of $\mathcal{R}$ and $a_\iota \subseteq \chi$ we have either $r_t \subseteq (\kappa :_R \chi)$ or $a_\iota \subseteq \kappa$ where $(\kappa :_R \chi) = \{ r_t\colon\ r_t\chi \subseteq \kappa,\ r_t$ fz-singleton of $\mathcal{R})$ [1]. Hasan, studied essential fz-submodules and introduce the term of closed fz-submodules as follows: Letχ be a fz-module and κ be a fz-submodule of χ, κ is siad closed (shortly $\kappa \leq_c \chi$), if κ has no proper essential, that is $\kappa \leq_e \rho \leq \chi$, then $\kappa = \rho$. In this paper, we introduce a new class of fz-submodules named z-essential fz-submodules and z-closed fz-submodule $\mathcal{R}$-fzmodule M. In the first section, we shall give some concepts and properties of fz-sets and fz-modules. In the second section, we introduce the notion of z-essential fz-submodule. Also we give several properties abut this concept [2]. In section three we define and examples of the notion z-closed fz-submodule and covers basic properties and existence of this concept.

Throughout this note all rings will have an identity, let χ is fuzzy module of an $\mathcal{R}$-module, M denoted by ($\mathcal{R}$-fzmodule M), and shortly fuzzy set, fuzzy submodule and fuzzy module is fz-set, fz-submodule and fz-module.

1.Preliminaries

This section, contains some definitions and properties of fz-sets, fz-modules and fz-submodules, which will used in the next sections.

Definition 1.1 :

Remember that T a nonempty set and let I will denote closed unit interval [0,1] of the real line (real numbers). A function μ: T $\longrightarrow$ I is a fz-set χ in T (a fz-subset χ of T) [3], [4].

Definition 1.2 :

Let x_ℓ : T $\longrightarrow$ I be a fz-set in T, $x \in$ T, $\ell \in [0,1]$, defined by:

$$x_\ell = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \quad \forall\ y \in \mathrm{T}$$

Then x_ℓ is named a fz-singleton.

If $x = 0$ and $\ell = 1$ then :

$$0_1(y) = \begin{cases} 1 & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}$$

Definition 1.3:

Let κ, ϱ be fz-sets in T, then :

1. κ = ϱ iff $\kappa(x) = ϱ(x)$, $\forall\, x \in$ T.
2. $\kappa \subseteq ϱ$ iff $\kappa(x) \leq ϱ(x)$, $\forall\, x \in$ T
3. $x_\ell \subseteq \nu$ iff x_ℓ(y) $\leq \kappa$ (y), $\forall$ y $\in$ T and if $\ell > 0$, then $\kappa(x) \geq \ell$. Thus $x_\ell \subseteq \kappa$ ($x \in \kappa_\ell$), (that is $x \in \kappa_\ell$ iff $x_\ell \subseteq \kappa$).

Definition 1.4:

Let κ, ϱ be fz-sets in T, then:

1.$(\kappa \cup \rho)(x)$ = max $\{(\kappa(\mathrm{x}), \rho(x)\}$, $\forall\, x \in$ T.

2.$(\kappa \cap \rho)(x) = \min\{(\kappa(x), \rho(x)\}$, $\forall\, x \in$ T.

$\kappa \cup \rho$ and $\kappa \cap \rho$ are fz-sets in T.

Definition 1.5:

Let κ be a fz-set in T, $\forall$ t $\in$ [0,1], the set κ_{t}= { $x \in$ T, $\kappa(x) \geq$ t} is named a level sub-set of κ.

Remark 1.6 :

Assume κ, ρ are fz-subsets of a set T, then:

1. $(\kappa \cap \rho)_t = \kappa_t \cap \rho_t$ for any t $\in$ [0,1] .
2. $(\kappa \cup \rho)_t = \kappa_t \cup \rho_t$ for any t $\in$ [0,1] .
3. $\kappa = \rho$ iff $\kappa_t = \rho_t$, $\forall$ t $\in$ [0,1].

Definition 1.7 :

Assuming f: M $\longrightarrow$ N and κ be fz-set in M, denoted by $f(\kappa)$ the image of κ is the fz-set in N defined by:

$$f(\kappa)(\mathrm{y}) = \begin{cases} sup\{\kappa(z) \mid z \in f^{-1}(y)\}\ if & f^{-1}(y) \neq \emptyset,\ for\ each\ y \in N \\ 0 & o.w \end{cases}$$

where $f^{-1}(y) = \{x : f(x) = y\}$

and let ρ be a fz-set in N, then the inverse image of ρ, denoted by $f^{-1}(\rho)$ is the fz-set in M defined by : $f^{-1}(\rho)(x) = \rho(f(x))$, $\forall\, x \in$ M.

Definition 1.8 : If f: M $\longrightarrow$ M~ be any mapping . A fz-subset ν of M is named f-invariant if $\kappa(x) = \kappa$ (y), then $f(x) = f$ (y), for all x, y $\in$ M [5].

Proposition 1.9:

Suppose that is a function defined on a set M, $\sigma_1\ and \sigma_2$ fz-subsets of M $\upsilon_1\ and\ \upsilon_2$ fz-subset of f(M). The consequently are true:

1. $\sigma_1 \subseteq f^{-1}(f(\sigma_1))$.
2. $\sigma_1 = f^{-1}(f(\sigma_1))$, whenever σ_1 is f-invariant.
3. $f(f^{-1}(\upsilon_1)) = \upsilon_1$.
4. If $\sigma_1 \subseteq \sigma_2$, then $f(\sigma_1) \subseteq f(\sigma_2)$.
5. If $\upsilon_1 \subseteq \upsilon_2$,then $f^{-1}(\upsilon_1) \subseteq f^{-1}(\upsilon_2)$.

Definition 1.10:

A fz-set χ of an $\mathcal{R}-$ module M is named fz-module ($\mathcal{R}$-fzmodule M) if :

1. χ $(x-y) \geq \min \{\chi(x), \chi(y)\}, \forall x, y \in.$ M
2. χ $(r\chi) \geq$ X $(\chi), \forall \chi \in$ M and r $\in \mathcal{R}$.
3. χ (0) = 1.

Definition 1.11:

Let χ, ω be two an $\mathcal{R}$-fzmodule M (fz-module of an$\mathcal{R}$ -module M). ω is named a fz-submodule of χ if $\omega \subseteq \chi$.

Definition 1.1:

If κ is an fz-submodule of an $\mathcal{R}$-fzmodule M, then the submodule κ_t of M is called level submodule of M, where t $\in$ [0,1].

Proposition 1.13 :

Let ν be a fz-module in M, then ν_* is submodule of M.

Proposition 1.14:

Assume that χ and γ be $\mathcal{R}$-fzmodules M_1 and M_2 (respectively) and $f: \chi \longrightarrow \gamma$ be a fz-homomorphism.

If $\sigma$and ρ are fz-submodules of χ and γ (respectively), we have :

1. $f(\sigma)$ is fz-submodule of γ .
2. $f^{-1}(\rho)$ is fz-submodule of χ.

Definition 1.15:

Let $\boldsymbol{\chi}$, γ be $\mathcal{R}$-fzmodules M_1, M_2 individually, define $\boldsymbol{\chi} \oplus\gamma$: $\mathrm{M}_1 \oplus \mathrm{M}_2 \longrightarrow [0,1]$ $(\chi\oplus\gamma)(h, g)$ = min { $\chi(h), \gamma(g), \forall (h, g) \in \mathrm{M}_1 \oplus \mathrm{M}_2$}.

$\chi\oplus\gamma$ is named fz-external direct sum of $\boldsymbol{\chi}$ and γ .

Definition 1.16:

Suppose that κ and ρ be two fz-submodules of $\mathcal{R}$-fzmodule M. We define (κ: ρ) by: = $\{r_t : r_t$ is a fz-singleton of $\mathcal{R}$ such that $r_t \rho \subseteq \kappa\}$, the residual quotient of κ and ρ.

Definition 1.17:

Let χ be a fz-module such that : Z(χ) = {$x_t \subseteq \chi$: F-ann(x_t) is an essential fz-ideal of $\mathcal{R}$} is named a fz-singular submodule of χ. If Z(χ) = χ and χ is named non-singular if Z(χ) = 0_1.

Definition 1.18

A fz-submodule ω is named torsion fz-submodule iff for each fz-point $x_t \subseteq \omega$ with $t > 0$, there exist $\iota \in \mathcal{R}, \iota \neq 0$ such that $\iota(x_t) = 0_1$.

Definition 1.19:

Remember that χ be $\mathcal{R}$-fzmodule M is named cyclic if there exist $x_t \subseteq \chi$ such that $\chi(x_t)$, hence for each $m_s \subseteq \chi$, there exists a fz-singleton a_ℓ of $\mathcal{R}$ such that $m_s = a_\ell x_t$, where t, s ,$\ell \in$ [0,1].

Definition 1.20:

Let σ be a fz-module in M, then we define, $\sigma_* = \{x \in \mathrm{M}: \sigma(x) = 1 = \sigma(0_{\mathrm{M}})\}$.

2. Materials and Methods

This research adopts a formal mathematical methodology grounded in fuzzy module theory to investigate and generalize the concepts of -essential and -closed fuzzy submodules. The study builds upon foundational definitions and properties of fuzzy sets, fuzzy modules, and submodules as previously established in classical and fuzzy algebraic literature. The method involves precise axiomatic development, where key definitions such as fuzzy singleton, level subsets, torsion fuzzy submodules, and fuzzy homomorphisms are rigorously employed to construct a consistent logical framework. The process begins by extending the traditional notion of essential submodules to define -essential fuzzy submodules, using properties of singular submodules as a basis for this generalization. A similar extension is formulated for closed fuzzy submodules, leading to the definition of -closed fuzzy submodules. Each new concept is accompanied by propositions and theorems, which are mathematically proven using standard techniques in module theory, including set inclusion, invariance under mappings, and properties of cyclic and prime submodules. Examples are constructed to demonstrate both inclusion and exclusion conditions for these generalized forms, and counterexamples are introduced where classical results do not hold under the new framework. The methodology also includes an investigation into the behavior of -closed submodules under module homomorphisms, showing preservation or transformation conditions. Overall, this approach emphasizes logical derivation, structural analysis, and internal consistency, ensuring that the newly introduced classes of fuzzy submodules are robust, coherent, and meaningful within the broader context of fuzzy algebraic systems.

3. Results

Enas in, introduced the concept of μ^*-essential submodule of an $\mathcal{R}$-module M (where a submodule A of an $\mathcal{R}$-module K is named μ^*-essential if $A \cap B \neq 0$, for any non-trivial singular submodule B of K [6], [7], [8]. We shall fuzzify this concept in definition (2.1).

Definition 2.1:

If γ be $\mathcal{R}$-fzmodule M and σ fz-submodule of γ, σ is named z-essential fz-submodule, if every non-empty singular fz-submodule δ of γ, we have $\sigma \cap \delta \neq 0_1$ (borfily $\sigma \leq_{ze} \gamma$).

Now, we explain the relation between z-essential fz-submodules and the submodules σ_t , σ_* by the following propositions.

Proposition 2.2:

Assume that σ be a fz-submodule of χ, if $\sigma_t \leq_{ze} \chi_t$, $\forall\ t \in [0,1]$, then $\sigma \leq_{ze}\ \chi$.

Proof:

ρ_t submodule of χ_t. There exists Since σ_t is z-essential submodule of χ_t, then $\sigma_t \cap \rho_t \neq 0$. Hence $(\sigma \cap\ \rho)_t \neq\ (0_1)_t$, $\forall\ t \in [0, 1]$, by Remark (1.6)(1), so $\sigma \cap \rho \neq 0_1$; that is σ is z-essential fz-submodule of χ .

Proposition 2.3:

Let χ be $\mathcal{R}$-fzmodule M, and σ be a fz-submodule of χ, then σ z-essential fz-submodule iff σ_* z-essential submodule of χ_*

Proof:

(⇒) Impose σ_* be z-essential, we have to show that σ is z-essential fz-submodule.

Let κ be fz-submodule of χ, and κ nonempty singular fz-submodule then κ_* non-zero singular of $\chi_*$by [2, lemma (2.5.3)]. Since σ_* is a z-essential, then $\sigma_* \cap \rho_* \neq 0$, imply $(\sigma \cap \rho)_* \neq 0$, by Remark (1.6), so $\sigma \cap \kappa \neq 0_1$. Thus σ is a z-essential fz-submodule.

(⇐) To prove that $\sigma \leq_{ze_*} \chi_*$

Let κ be non-zero singular submodule of χ_*. To prove $\sigma_* \cap \kappa \neq 0$. Define ρ : M ⟶[0,1], by $\rho(a) = \begin{cases} 1 & if\ a\ \in \kappa \\ 0 & otherwise \end{cases}$

It is clear that $\rho \leq X$ and $\rho_* = \kappa$. Since κ is singular fz-submodule, we have by [2, Lemma(2.5.3)], ρ is singular fz-submodule and since σ is a z-essential, so $\sigma \cap \rho \neq 0_1$, then $(\sigma \cap \rho)_* \neq 0$, by Remark (1.6), $\sigma_* \cap \rho_* \neq 0$, imply that $\sigma_*$is a z-essential submodule [9], [10].

Remarks and Examples 2.4:

1. Certainly that z-essential fz-submodule is generalization of essential fz-submodule. There is a z-essential fz-submodule of χ which isnot essential in χ, for example:

Let M be Z_6 as Z_6-module and χ: : M ⟶[0,1], define by : $\chi(\alpha) = 1$, $\forall\ \alpha \in Z_6$

Let κ: M ⟶ [0,1], defined by: $\kappa(a) = \begin{cases} 1 & if\ a\ \in (3) \\ 0 & otherwise \end{cases}$, $\sigma(a) = \begin{cases} 1 & if\ a\ \in (2) \\ 0 & otherwise \end{cases}$

It is easy that κ, σ are fz-submodule and $\kappa_* = (3)$ $\sigma_* = (2)$, are z-essential in Z_6 which are not essential in Z_6 by [12, Remarks and Examples (3.1.3)]. Hence κ, σ are z-essential, but n't essential by Proposition (2.3).

2. Let M be Q be Z-module and $\chi : Q$ ⟶[0,1], define by : $\chi(\alpha) = 1$, $\forall\ \alpha \in Q$

It is easy that χ is a fz-module and χ_t = Q and Q is a z-essential in Q by [12, Remarks and Examples (3.1.2)], so χ is z-essential by Proposition. (2.3).

3. Let M be Z_6 as Z-module and χ: : M ⟶[0,1], define by : $\chi(\alpha) = 1$, $\forall\ \alpha \in Z_6$

Let ν: M ⟶ [0,1], defined by: $\nu(a) = \begin{cases} 1 & if\ a\ \in (3) \\ 0 & otherwise \end{cases}$, $\sigma(a) = \begin{cases} 1 & if\ a\ \in (2) \\ 0 & otherwise \end{cases}$

It is easy $\chi_* = Z_6$ and ν, σ are fz-submodules and $\nu_* = (3), \sigma_* = (2)$ which are not z-essential by [12, Remarks and Examples (3.1.3)]. Hence κ, σ are not z-essential,(See Proposition (2.3).

We provide sufficient requirements in this proposition for the z-essentia fz-submodules to make the essential fz-submodules [11], [12].

Proposition 2.5:

Let χ be a singular fz-module and σ a fz-submodule of χ, then $\sigma \leq_{ze} \chi\ iff\ \sigma \leq_e \chi$.

Proof:

It is easy.

Proposition 2.6:

Let χ be a torsion fz-module over commutative intger-domin $\mathcal{R}$ and σ fz-submodule of χ. Then $\sigma \leq_{ze} \chi$ iff $\sigma \leq_e \chi$.

Proof:

Since χ is a torsion fz-module, then by [10, Proposition(2.1.1)], χ_t torsion module, so by [13, p.31] and [12, Proposition (3.1.6)],we have the result by proposition (2.5).

Proposition 2.7:

Let χ be a prime fz-module of an R-module M with $Z(\chi) \neq 0_1$and σ be a fz-submodule of χ. Then $\sigma \leq_{ze} \chi$ iff $\sigma \leq_e \chi$.

Proof:

Assume that $\sigma \leq_{ze} \chi$, we have to show that χ is singular. Let $0_1 \neq x_1 \subseteq Z(\chi)$, then $\text{ann}(\chi_1) \leq_e \mathcal{R}$ and $0_1 \neq y_1 \subseteq \chi$. Since χ is prime fz-module, then $\text{ann}(\chi_1) = \text{ann}(y_1)$ and $y_1 \subseteq Z(\chi)$. Thus Z(M) = M, so we have $\sigma \leq_e \chi$ by proposition (2.5).

Next we give characterizations of z-essential fz-submodule

Proposition 2.8:

Assumig χ be $\mathcal{R}$-fzmodule Mand σ be a fz-submodule, then $\sigma \leq_{ze} \chi$ iff for any non-empty cyclic singular fz-submodule κ of χ, $\sigma \cap \kappa \neq 0_1$.

Proof:

Assume that κ non-empty cyclic singular fz-submodule and let $0_1 \neq x_t \subseteq \kappa$. By our assumption $0_1 \neq (x_t) \cap \sigma \subseteq \sigma \cap \kappa$. Hence $\sigma \cap \kappa \neq 0_1$

The converse is directly.

Proposition 2. 9:

Let χ be $\mathcal{R}$-fzmodule M and σ be a fz-submodule of χ, then $\sigma \leq_{ze} \chi$ iff $0_1 \neq (x_t) \subseteq \chi$ with $\mathcal{R}x_t$ singular has non-empty multiple in σ.

Proof:

Assume that $0_1 \neq (x_t) \subseteq \chi$ and $\mathcal{R}x_t$ singular fz-submodule of χ. By Proposition (2.8), $\mathcal{R}x_t \cap \sigma \neq 0_1$. Imply there is $r_t \subseteq \mathcal{R}$ such that $(rx)_t \subseteq \sigma$. The conversely is clear.

Remark 2.10:

Every fz-submodule of χ is is a z-essential fz-submodule in itself.

Proof:

Immediate by definition (2.1)..

Proposition 2.11:

Let χ be $\mathcal{R}$-fzmodule M, then

i-$\sigma_1 \leq_{ze} \rho_1$ and $\sigma_2 \leq_{ze} \rho_2$, then $\sigma_1 \cap \sigma_2 \leq_{ze} \rho_1 \cap \rho_2$.

ii-$\sigma_1 \leq_{ze} \chi$ and $\sigma_2 \leq_{ze} \chi$, then $\sigma_1 \cap \sigma_2 \leq_{ze} \chi$.

Proof:

i- Assume that η non-empty singular fz-submodule such that $\eta \subseteq \rho_1 \cap \rho_2$.Since $\sigma_2 \leq_{ze} \rho_2$, then $\sigma_2 \cap \eta \neq 0_1$, imply $\sigma_2 \cap \eta$ non-empty singular fz-submodule such that $\sigma_2 \cap \eta \subseteq \rho_1$.As $\sigma_1 \leq_{ze} \rho_1$, hence $\sigma_1 \cap (\sigma_2 \cap \eta) \neq 0_1$, so $(\sigma_1 \cap \sigma_2) \cap \eta \neq 0_1$ thus $\sigma_1 \cap \sigma_2 \leq_{ze} \rho_1 \cap \rho_2$.

ii- It is clear by Remark (2.10), and Proposition (2.11-i).

Proposition 2.12:

Impose χ be $\mathcal{R}$-fzmodule M,and $\kappa \leq \sigma \leq \chi$. If $\kappa \leq_{ze} \sigma$ and $\sigma \leq_{ze} \chi$, then $\kappa \leq_{ze} \chi$.

Proof:

$\kappa \leq_{ze} \sigma$ and $\sigma \leq_{ze} \chi$ with η non-empty singular fz-submodule . Since $\sigma \leq_{ze} \chi$, we have $\sigma \cap \eta \neq 0_1$ and since $\kappa \leq_{ze} \sigma$, we have $(\eta \cap \sigma) \cap \kappa \neq 0_1$, that is $\eta \cap \kappa \neq 0_1$, imply $\kappa \leq_{ze} \chi$.

Remark 2.12:

Suppose that σ_1, σ_2, ρ_1 and ρ_2 be fz-submodules of $\mathcal{R}$-fzmodule M if $\sigma_1 \leq_{ze} \rho_1$ and $\sigma_2 \leq_{ze} \rho_2$, then it's n't necessary that $(\sigma_1 + \sigma_2) \leq_{ze} (\rho_1 + \rho_2)$ as the example below:

Example:

Let M = Z-module $Z \oplus Z_2$, let defined χ : M ⟶ [0,1], by :

χ(a ,b) = 1, ∀ (a, b) ∈ $Z \oplus Z_2$

Let σ: M ⟶ [0,1] , ρ : M ⟶ [0,1] , σ_1=σ_2 (a ,b) = $\begin{cases} 1 \text{ if } (a,b) \in (2,\overline{0})Z \cong Z \\ 0 \quad\quad \text{otherwise} \end{cases}$

ρ_1 =(a ,b) = $\begin{cases} 1 \text{ if } (a,b) \in (1,\overline{0})Z \cong Z \\ 0 \quad\quad \text{otherwise} \end{cases}$, ρ_2(a, b) = $\begin{cases} 1 \text{ if } (a,b) \in (1,\overline{1})Z \\ 0 \quad\quad \text{otherwise} \end{cases}$

It is easily σ_1, σ_2, ρ_1 and ρ_2 be fz-submodules of χ. One easily show that $\sigma_1 \leq_{ze} \rho_1$ and $\sigma_2 \leq_{ze} \rho_2$. But $(\sigma_1 + \sigma_2) \nleq_{ze} (\rho_1 + \rho_2)$. Because there exist μ: M ⟶ [0,1], define by:

$\mu(a,b) = \begin{cases} 1 \text{ if } (a,b) \in 0 \oplus Z_2 \\ 0 \quad\quad \text{otherwise} \end{cases}$. Where μ is non-empty singular fz-submodule of ρ_1+ρ_2

Such that(ρ_1+ρ_2) $\cap \mu = (0_1, 0_1)$.

4. Discussion

We spoke in this section about the notion of z-closed fz-submodules as a generalization of closed fz-submodule, also we shall establish the basic properties about this concept is given [13], [14].

Definition 3.1:

Let κ be a fz-submodule of χ. κ is named z-closed in χ (briefly $\kappa \leq_{zc} \chi$), if has no proper z-essential in χ.

Now, we shall discuss the relationship between z-closed fz-submodules and it's level submodules.

Let (*) means the following : For a fz-module $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$, $\boldsymbol{\nu}$ be non-empty fz-submodules of **X**, if $\boldsymbol{\rho}_* \subseteq \boldsymbol{\nu}_*$ implies that $\boldsymbol{\rho} \subseteq \boldsymbol{\nu}$.

Proposition 3.2:

Assume that χ be $\mathcal{R}$-fzmodule M and σ fz-submodule of χ, then $\sigma \leq_{zc} \chi$ iff $\sigma_* \leq_{zc} \chi_*$.

Proof:

Assume that $\sigma_* \leq_{ze} N \leq \chi$. Our immediate $\sigma_* = N$. Let ρ : M ⟶[0,1], by: ρ(a) = $\begin{cases} 1 & if\ a \in N \\ 0 & otherwise \end{cases}$

Clearly ρ fz-submodule of χ and ρ_*= N, hence $\sigma_* \subseteq \rho_* = N$, and by condition (*) $\sigma \subseteq \rho$.but $\sigma_* \leq_{ze} N = \rho_*$, then by Proposition (2.3), $\sigma \leq_{ze} \rho$, since σ is a z-closed fz-submodule in χ ; therefore $\sigma = \rho$, so $\sigma_* = N = \rho_*$.Thus $\sigma_* = N$.

Conversely $\sigma_* \leq_{zc} \chi_*$. Aim is to show $\sigma \leq_{zc} \chi$.

Assume that $\sigma \leq_{ze} \rho \leq \chi$ we must prove $\sigma = \rho$. Since $\sigma \leq_{ze} \rho$, then $\sigma_* \leq_{ze} \rho_*$by proposition (2.3). But σ_* is a z-closed submodule in χ_*, so $\sigma_* = \rho_*$,imply $\sigma = \rho$ by condition (*).

Remarks 3.3:

1. Let σ be a fz-submodule of a fz-module χ . If $\sigma_t \leq_{zc} \chi_t$, ∀ t ∈ [0,1], then $\sigma \leq_{zc} \chi$.

Proof:

Suppose there exists ϱ singular fz-submodule such that $\nu \leq_{ze}$ ρ. Then $\sigma_t \leq_e$ ρ_t ∀ t ∈ [0,1]. But σ_t is z-closed submodule of χ_t, ∀ t ∈ [0,1], so that $\sigma_t = \rho_t$, ∀ t ∈ [0,1]. Hence by Remark (1.6)(3), σ = ϱ. Thus $\sigma \leq_{zc} \chi$.

2. Let χ be $\mathcal{R}$-fzmodule M such that χ(a) = 1. Let N ≤ M and σ : M ⟶ [0,1] defined by:

σ(x) = $\begin{cases} 1 & \text{if } x \in N \\ c & \text{otherwise} \end{cases}$, where $0 \leq c < 1$. If N $\leq_{zc}$ M , then $\sigma \leq_{zc} \chi$.

Proof:

Let t ∈ [0,1]. If t = 0, then σ_t= M, if t > c, then ν_t = N and if t < c implies that σ_t= M. Hence $\sigma_t \leq_{zc} \chi_t$ = M, ∀ t ∈ [0,1] and so by part (1), $\sigma \leq_{zc} \chi$.

Remarks and Examples 3.4:

1- Let M = Z_6 as Z-module, let χ: : M ⟶[0,1], define by : $\chi(\alpha) = 1, \forall \alpha \in Z_6$

Let ν: M ⟶ [0,1], defined by: $\nu(a) = \begin{cases} 1 & if\ a \in (3) \\ 0 & otherwise \end{cases}$, $\sigma(a) = \begin{cases} 1 & if\ a \in (2) \\ 0 & otherwise \end{cases}$

It is easy $\chi_* = Z_6$ and ν, σ are fz-submodules and $\nu_* = (3), \sigma_* = (2)$ which are z-closed by [12, Remarks and Examples (3.1.14)]. Hence κ, σ are z-closed fz-submodule,(See Proposition (3.2).

2. Let M be Z_4 as Z-module and χ: : M ⟶[0,1], define by : $\chi(\alpha) = 1, \forall \alpha \in Z_4$

Let ν: M ⟶ [0,1], defined by: $\sigma(a) = \begin{cases} 1 & if\ a \in (2) \\ 0 & otherwise \end{cases}$

It is easy σ is a fz-submodules and $\sigma_* = (2)$ which not z-closed in Z_4 by [12, Remarks and Examples (3.1.14)]. Hence , σ is not z-closed,(See Proposition (3.2).

3-Every z-closed fz-submodule is closed fz-submodule, but the opposite is untrue, like the example below:

Let M be Z_6 as Z_6-module and χ: : M ⟶[0,1], define by : $\chi(\alpha) = 1, \forall \alpha \in Z_6$

Let κ: M ⟶ [0,1], defined by: $\kappa(a) = \begin{cases} 1 & if\ a \in (3) \\ 0 & otherwise \end{cases}$, $\sigma(a) = \begin{cases} 1 & if\ a \in (2) \\ 0 & otherwise \end{cases}$

It is easy that κ, σ are fz-submodule and $\sigma \oplus \kappa = \chi$, hence σ, κ are closed fz-submodules (see [2], proposition(2.1.15)), but not z-closed fz-submodule since $\nu_* = (3), \sigma_* = (2)$ are not z-closed (see [12., Remarks and Examples (3.1.14-3)), so by proposition(2.3). κ, σ are not z-closed fz-submodules.

4- If χ be a singular fz-module and let σ a fz-submodule of χ, we have $\sigma \leq_c \chi\ iff\ \sigma \leq_{zc} \chi$.

5- If χ be a torsion fz-module over commutative intger-domin R and σ fz-submodule of χ. Then $\sigma \leq_{zc} \chi$ iff $\sigma \leq_c \chi$.

6- Let χ be a prime $\mathcal{R}$-fzmodule M with $Z(\chi) \neq 0_1$, let σ be a fz-submodule of χ. Then $\sigma \leq_{zc} \chi$ iff $\sigma \leq_c \chi$.

7-Remember that every direct summand of fz-module χ is closed. But in case z-closed there is not necessary. (See Remark and Examples (3))

Remark 3.5:

In general any two z-closed fz-submodules of χ intersection are n't once again z-closed fz-submodule like the example below:

Example:

Let M = Z-module $Z \oplus Z_2$ and let:

$\chi(a, b) = 1, \forall (a, b) \in Z \oplus Z_2$

Defined σ: M ⟶ [0,1] , ρ : M ⟶ [0,1] ,by :

$\sigma(a, b) = \begin{cases} 1 \text{ if } (a,b) \in (1,\overline{0})Z \cong Z \\ 0 \qquad\qquad \text{otherwise} \end{cases}$, $\rho(a, b) = \begin{cases} 1 \text{ if } (a,b) \in (1,\overline{1})Z \\ 0 \qquad\qquad \text{otherwise} \end{cases}$

clearly σ, ρ be fz-submodules of χ and $\sigma_* = (1,\overline{0})Z \cong Z$,$\rho_* = (1,\overline{1})Z$. because $\sigma_* = 0 \oplus Z_2$, the only one singular submodule of χ_* and intersection zero with the σ_*, then $\sigma_* \leq_{zc} \chi_*$, so $\sigma \leq_{zc} \chi$ by proposition (3.2). Similarly $\rho \leq_{zc} \chi$.

$(\sigma \cap \rho)(a,b) = \min\{\sigma(a), \rho(b)\}\ \forall (a, b) \in Z \oplus Z_2$

$(\sigma \cap \rho)(a,b) = \begin{cases} 1 \text{ if } (a,b) \in (2,\overline{0})Z \\ 0 \qquad\qquad \text{otherwise} \end{cases}$

$(\sigma \cap \rho) \leq_e \sigma$, but $\sigma \cap \rho \neq \sigma$. Thus $\sigma \cap \rho$ is not z-closed fz-submodule of χ.

Next give the basic properties of z-closed fz-submodules

Proposition 3.6:

Assume that χ be $\mathcal{R}$-fzmodule M. If $\sigma \leq_{zc} \chi$, then $\rho/\sigma \leq_{ze} \chi/\sigma$ whenever $\sigma \leq \rho \leq_{ze} \chi$.

Proof:

Let $\sigma \leq \rho \leq_{ze} \chi$. Then $\sigma_* \leq \rho_*$ by Proposition (1.13) and $\rho_* \leq_{ze} \chi_*$ by Proposition (2.2), therefore $\sigma_* \leq \rho_* \leq_{ze} \chi_*$. But $\sigma \leq_{zc} \chi$ implies that $\sigma_* \leq_{zc} \chi_*$ by proposition (3.2). Hence $\rho_*/\sigma_* \leq_{ze} \chi_*/\sigma_*$ see [12,Proposition(3.1.15)] that is $(\rho/\sigma)_* \leq_{ze} (\chi/\sigma)_*$ by[14, Proposition (1.22)] and $\rho / \sigma \leq_{ze} \chi / \sigma$ by Proposition (3.2) .

Now, we study the homomorphism image of z-closed fz-submodule.

Proposition 3.7:

Let χ and γ be two $\mathcal{R}$-fzmodule M_1,M_2 respectively, let $f : \chi \longrightarrow \gamma$ be a fz-epimorphism. If σ is a z-closed fz-submodule of χ such that χ andγ are f-invariant, then $f(\sigma)$ is z-closed fz-submodule of γ.

Proof:

Since σ fz-submodule of χ, then $f\ (\sigma)$ is a fz-submodule of γ see Proposition (1.14)(1). Suppose that σ is a z-closed fz-submodule in χ and $f(\sigma) \leq_{ze} \rho$, we get $f^{-1}(f(\sigma)) \leq_{ze} f^{-1}(\rho)$, where $f^{-1}(\rho)$ is a fz-submodule of χ, by Proposition (1.14)(2) and since σ is f-invariant, so $f^{-1}\big(f(\sigma)\big) = \sigma$, that is $\sigma \leq_{ze} f^{-1}(\rho)$. But σ is a z-closed fz-submodule, then $\sigma = f^{-1}(\rho)$ and since f is an epimorphism, so $f(\sigma) = \rho$. Hence $f(\sigma)$ is a z-closed fz-submodule in γ [15].

One can easily prove the following corollary.

Corollary 3.8:

z-closed fz-submodule is closed fz-submodule under isomorphism.

Corollary 3.9:

Let χ be $\mathcal{R}$-fzmodule M, if $\sigma \leq_{zc} \chi$ and $\sigma \leq \rho \leq \chi$, then $\sigma \leq_{zc} \rho$.

Proof:

Assume that $\sigma \leq_{ze} \eta$, where $\eta \leq \rho$. It clear that $\eta \leq \chi$, imply that $\sigma = \eta$ since $\sigma \leq_{zc} \chi$. Thus $\sigma \leq_{zc} \rho$.

Recall that a F-module μ is a chained, if for each F-sub module $\nu\ and\ \rho$ of μ either $\nu \leq \rho$ or $\rho \leq \nu$ [15].

Proposition 3.10:

Suppose that χ be $\mathcal{R}$-fzmodule M and $\sigma \leq \rho \leq \chi$. If $\sigma \leq_{zc} \rho$ and $\rho \leq_{zc} \chi$, then $\sigma \leq_{zc} \chi$.

Proof:

Since, $\sigma \leq_{zc} \rho$ and $\rho \leq_{zc} \chi$, we certainly have by Proposition (3.2) σ_* is a z-closed in ρ_* and ρ_* is a z-closed in χ_* imply σ_* is z-closed in χ_* see [12, Proposition(3.1.19)]. Therefore by Proposition (3.2), $\sigma \leq_{zc} \chi$.

Proposition 3.11:

Impose χ be $\mathcal{R}$-fzmodule M, let σ, ρ be fz-submodules with $\sigma \leq \rho \leq \chi$. If $\sigma \leq_{zc} \chi$, then $\sigma \leq_{zc} \rho$.

Proof:

Assume that $\sigma \leq_{zc} \kappa \leq \rho \leq \chi$. But $\sigma \leq_{zc} \chi$, imply that $\sigma = \kappa$. Thus $\sigma \leq_{zc} \rho$.

Proposition 3.12:

When χ be $\mathcal{R}$-fzmodule M and σ be a non-trivial fz-submodule of χ, then there exists z-closed fz-submodule ρ in χ such that $\sigma \leq_{ze} \rho$. ρ is called a closure of σ.

Proof:

Let S = { $\eta : \eta \leq \chi; \sigma \leq_{zc} \eta$ }. Clearly S is non-empty set. S has maximal element say ρ by Proposition [2, Proposition (2.1.12)]. To prove that $\rho \leq_{zc} \chi$. Assuming there exists a fz-submodule κ of χ, we have $\rho \leq_{ze} \kappa$. Since $\sigma \leq_{ze} \rho \leq_{ze} \kappa$, so by Proposition (2.11), we have $\sigma \leq_{ze} \kappa$. But this is a contradicts the maximality of ρ. Thus $\rho = \kappa$, hence $\rho \leq_{zc} \chi$ with $\sigma \leq_{ze} \rho$.

Theorem 3.13:

Assuming $\{\sigma_\alpha\}$, $\{\chi_\alpha\}$ be family of $\mathcal{R}$-fzmodules M, we have $\sigma_\alpha \leq_{zc} \chi_\alpha$, for each α. Then $\oplus\sigma_\alpha \leq_{zc} \oplus\chi_\alpha$, $\alpha \in \Lambda$ (any index set) .

Proof:

Suppose $\oplus\sigma_\alpha \leq_{zc} \rho$, where ρ is a fz-submodule of $\oplus\chi_\alpha$ For any $\alpha_i \in \Lambda$, $X_{\alpha_i} \leq_{ze} X_{\alpha_i}$. Hence by Proposition (2.11-i) ,$\sigma_{\alpha_i} = \oplus\sigma_\alpha \cap \chi_{\alpha_i}$ is a z-essential in $\rho \cap \chi_{\alpha_i} \leq \chi_{\alpha_i}$. Since σ_{α_i} is z-closed in χ_{α_i} by assumption, hence $\sigma_{\alpha_i} = \rho \cap \chi_{\alpha_i}$. But $\oplus\ \sigma_\alpha \leq \rho$, hence $\rho\ \cap \sigma_{\alpha_i} = \sigma_{\alpha_i}\ \forall\ \alpha_i \in \Lambda$. As $\rho\ \cap \big(\oplus\sigma_{\alpha_i} \cap \chi_{\alpha_i}\big) = \rho \cap \chi_{\alpha_i} = \sigma_{\alpha_i}$. It follow that $\rho \subseteq \oplus_{\alpha_i}\sigma_{\alpha_i}$, because $x_t \subseteq \rho \leq \oplus\chi_{\alpha_i}$, $(x_t)_{\alpha_i}$ the α_i-component of x_t is in χ_{α_i}. Thus $(x_t)_{\alpha_i} \subseteq \rho \cap \chi_{\alpha_i} = \sigma_{\alpha_i}$ and this implies $(x_t)_{\alpha_i} \subseteq \sigma_{\alpha_i}$ for any $\alpha_i \in \Lambda$. Thus $\rho \leq \oplus_{\alpha_i}\sigma_{\alpha_i}$; that is $\rho = \oplus\sigma_\alpha$ and σ_α is a z-closed in $\oplus\chi_\alpha$.

5. Conclusion

In conclusion, this study introduced and rigorously explored the concepts of -essential and -closed fuzzy submodules as significant generalizations of essential and closed fuzzy submodules within -fuzzy modules. The key findings demonstrate that while every -essential fuzzy submodule encompasses the structure of essential submodules, it extends beyond their classical limitations by encompassing singular submodules under broader conditions. Additionally, -closed fuzzy submodules were shown to preserve the closure property under specific homomorphic mappings, though the converse does not always hold—highlighting a structural asymmetry that invites further scrutiny. These results not only provide new characterizations for fuzzy submodule interactions but also offer a framework for analyzing submodule behavior in more complex algebraic systems. The implications of these findings are substantial for the advancement of fuzzy module theory, especially in enhancing our understanding of module hierarchies, invariance properties, and the algebraic underpinnings of partial membership structures. Future research may delve into the categorical relationships between -closed submodules and their classical counterparts, examine computational algorithms for identifying such submodules in large fuzzy systems, and investigate the applicability of these concepts within applied domains such as fuzzy logic control, data clustering, and algebraic cryptography.

REFERENCES

[1] R. H. Jari, Prime Fuzzy Submodules and Prime Fuzzy Modules, M.Sc. thesis, University of Baghdad, 2001.

[2] H. K. Marhoon, Fuzzy Closed Submodules and Fuzzy W-closed Submodules with the Some of Their Generalization, Ph.D. dissertation, University of Baghdad, 2020.

[3] L. A. Zadeh, "Fuzzy Sets," Information and Control, vol. 8, pp. 338–353, 1965.

[4] M. M. Zahedi, "On L-Fuzzy Residual Quotient Module and P. Primary Submodule," Fuzzy Sets and Systems, vol. 51, pp. 333–344, 1992.

[5] L. Martinez, "Fuzzy Module Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings," J. Fuzzy Math., vol. 4, pp. 843–857, 1996.

[6] Z. Yue, "Prime L-Fuzzy Ideals and Primary L-Fuzzy Ideals," Fuzzy Sets and Systems, vol. 27, pp. 345–350, 1988.

[7] R. Kumar, "Fuzzy Semi-primary Ideals of Rings," Fuzzy Sets and Systems, vol. 42, pp. 263–272, 1991.

[8] R. Kumar, S. K. Bhambir and P. Kumar, "Fuzzy Submodule of Some Analogous and Deviation," Fuzzy Sets and Systems, vol. 70, pp. 125–130, 1995.

[9] M. Mashinchi and M. M. Zahedi, "On L-Fuzzy Primary Submodule," Fuzzy Sets and Systems, vol. 49, pp. 231–236, 1992.

[10] E. M. Kamil, μ-Lifting and μ*-Extending Modules, Ph.D. dissertation, University of Baghdad, 2019.

[11] J. T. Kight, Commutative Algebra, Cambridge University Press, 1971.

[12] M. A. H. Maysoun and H. Y. K. Hatam, "Maximal Fuzzy Submodules," Iraq Journal of Science, vol. 61, no. 5, pp. 1164–1172, 2020.

[13] S. B. Semeen, "Chained Fuzzy Modules," Ibn Al-Haitham Journal for Pure and Applied Sciences, vol. 23, no. 2, 2010.

[14] L. Martinez, "Fuzzy Module Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings," J. Fuzzy Math., vol. 4, pp. 843–857, 1996.

[15] A. A. Qaid, Some Results on Fuzzy Modules, M.Sc. thesis, University of Baghdad, 1991.