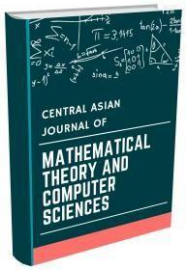

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Article

Existence, Uniqueness, and Numerical Solutions for 2D Volterra-Fredholm Equations with Singular Kernels

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Abstract: Vaultra-Fredhom Integrated integrated integrated equations and non-linear cores have extensive applications in modelling complex physical problems such as heat transfer, fluid mobility and image processing. Because of their special complications, these equations are difficult to solve analytically and require effective and stable numerical methods to get an accurate solution. The main objective of this research is to investigate the existence and specificity properties of the two-dimensional Volterra peace-to-life solution. In this connection, the taum method was used as an accurate numerical technique to solve both linear and non-linear equations. In this study, integrated equations were converted to algebraic systems, and their numerical solutions were achieved effectively using base functions and a series of appropriate mats. The results suggest that the taum method is able to show high stability in producing accurate solutions for complex equations under different border conditions and solving equations with a cynical core. This research is used especially in engineering and physics in non -linear and complex problems.

Keywords: Integral Equations, Volterra-Fredholm, Singular Kernels, Tau Method, Numerical Methods

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1. Introduction

Volterra-Fredholm, especially in two-dimensional and other forms, plays an important role in modeling complex scientific problems such as heat transfer, fluid mobility, biological models, and imaging [1]. These equations, which combine memory-based structures (Volterra) with stable, domain-bound operators (Fredholm), include an unknown work under the integrated symbol so that they can analyze and calculate their theoretical and numerical challenges [2]. When these equations have a unique core, the solution becomes even more important for the development of precise analysis of behavior and stable numerical methods [3]. The equations with an eccentric core are often seen in analyzing actual physical applications such as thin film phenomena, magnetic flow and wave integration, and their solutions can provide deep insight into non-larynx phenomena[4].

Because of their special mathematical properties, the mixture of existence and unique principles of accurate and effective numerical techniques to solve this class of problems [5]. Thus, a fundamental first step to promote both theoretical and practical knowledge in integrated equations [6] examines existence and uniqueness and offer a strong technique for numerical solutions to these equations. Solving a two-dimensional Voltra-Fredhom integrated equation with a single core gives several difficulties [7]. On the one hand, ecosystem's existence in the core reduces the effectiveness of the traditional numerical

approach to convergence and stability [8]. On the other hand, the calculation of the two - dimensional structural method increases dimensions and complexity. In addition, these equations are often unhealthy and cannot be resolved directly using the correct adjustment of regularity or a minor approach [9].

In these situations, another crucial barrier to provide a more accurate answer [10] is a universal and extendable structure for theoretical examination of survival and uniqueness. Several research on the analysis and numerical solution of two-dimensional Voltra-Fredhom integrated equations have lately attracted attention; when the topic from each particular viewpoint [11], [12]. These studies have laid a basis for developing more accurate methods [13], and offer different theoretical and numerical approaches. For example, Micula [14] suggested a numerical method based on the estimates of sequential value-type to solve second-to-dimensional Vaultra-Fredhome equations. This method achieved fast convergence compared to the classic Picard method using a customized Cubcher formula and a sophisticated recurrent structure. In that study, existence and specificity under specific mathematical conditions were proven with theoretical error estimates and convergence analysis. Numerical results demonstrated high accuracy and remarkable stability of the proposed method under different conditions [14].

On the other hand, Shalangwa et al. [15] used multidimensional Voltra-Fredeom-Predom-Fedhome integrated equations based on Bernstein functions. By converting the integrated equation to an algebraic system and applying the Gauss elimination algorithm, he developed a numerical model in many examples and faster convergence for an accurate solution as the projection parameter NN increased. These results confirmed the effectiveness of the algorithm proposed to improve accuracy and reduce calculation costs [15].

In another study [16], introduced a hybrid method by combining numerical regularization and the Chebyshev Wavelet algorithm to solve a system of integrated equations for the first and second CRI. In this method, the first type of sickness type Equation was previously converted into a well-grounded second time, and then a stable numeric algorithm was designed using chebashev waves. Complete confirmation through theoretical convergence analysis and numerical examples have shown that the proposed algorithm not only produces very accurate solutions, but also effectively handles the equations with eccentricity or [16].

Despite this effort, a clear difference remains in the strict theoretical examination of existence and the uniqueness of two-dimensional Voltra-peace equations with a clear core [17]. In addition, most of the existing numerical methods(are designed for simple cases or continuous cores and lack stability when facing eccentricities and two-dimensional structures. This research difference emphasises the requirement for a comprehensive study with a common theoretical and numerical approach.The main objective of this study is to examine the existence and specific properties of the solution with eccentric cores for two-dimensional Volterra-Fredholm integration equations, as well as develop and analyse more stable and precise numerical methods to solve them. At the same time, the innovation of this research focuses on the development of algorithms that are capable of solving the equations with eccentric cores accurately. Under the following section study, the model's mathematical structure, theoretical analysis of existence and specificity, design and implementation of numerical methods and execution of numerical examples present the results obtained from the execution of examples.

2. Tau Method for Linear Integral-Differential Equations

2.1. Problem

Volterra and Fredholm Types of Integrated-Power Equations Play an important role in physical and engineering problems and are widely used to model phenomena such as heat transfer, wave spread and biological events. These equations are usually presented as

either non-linear or linear equations and the combination of integrated operators. In this section, the linear integrated and unfair equations will be examined as follows:

$$Ly(x) := Dy(x) + \lambda \int_a^b K_1(x,t)y(t)\,dt + \int_x^a K_2(x,t)y(t)\,dt = f(x) \quad (1)$$

In this equation:

a. $\boldsymbol{Dy(x)}$ represents a differential operator, typically involving various derivatives of the function $\mathbf{y(x)}$.

b. $\boldsymbol{K_1(x,t)}$ and $\boldsymbol{K_2(x,t)}$ are the kernels of the integral equations, appearing in two separate integrals.

c. $\boldsymbol{\lambda}$ is a constant that acts as a weighting parameter in the equation.

d. $\mathbf{f(x)}$ is a given function on the right-hand side of the equation, and the objective is to find the solution $\mathbf{y(x)}$ using numerical methods.

This equation can be challenging to compute, particularly in the context of an eccentric or pathological core, as it involves both integrals and derivatives. Consequently, addressing such issues necessitates the application of appropriate numerical methods.

This study use the numerical technique of the taum method to resolve linear integral equations. This technique enables the treatment of the equation's solution as a sequence, so converting it into a flexible system of algebraic equations. In this context, the provided function F(x), together with the kernels $K^1(x, t)$ and $K^2(x, t)$, must be accurately computed using polynomial or other basis functions to facilitate the relaxation of the equation. Typically originating from the Tau chain, these approximations facilitate the numerical problem-solving process..

2.2. Theorems

This section will handle integrated and unwanted equations by covering significant concepts and results on the taum method. This method is based on the use of three basic simple matrixes, usually used to solve both intended and undesirable equations..

2.2.1. Basis Matrices

The three simple matrices used in the Tau method are as follows:

1. The matrix $\boldsymbol{\mu}$, which is defined as:

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad (2)$$

This matrix is used to model the behavior of the unknown function **y(x)y(x)** in the Tau method.

2. The matrix $\boldsymbol{\eta}$, which is defined as:

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad (3)$$

The matrix η is used for performing calculations related to derivatives and for transforming the equation into a system of algebraic equations.

3. The matrix ι, which is used for integration operations:

$$\iota = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad (4)$$

This matrix is particularly used when converting integrals into numerical forms and solving equations.

2.2.2. Differential Operator

To apply the differential operator to the unknown function $y(x)$, the matrices η and μ are used. In general, the differentiation is performed as follows:

$$\frac{d}{dx}y(x) = a^T \eta X \quad (5)$$

Here, $\boldsymbol{a^T}$ is a vector of coefficients, and X is a vector of variables, which is expressed as a series of basis functions. This formula is used for numerical computations of the derivatives of the unknown function $y(x)$.

2.2.3 Integration

To convert the integral operator into a numerical form, the matrix μ is used. Specifically, the integration operation is performed as follows:

$$\int y(x)\, dx = a^T \iota X \quad (6)$$

This formula is used to convert the integrals present in the integral equations into a computable format. Here, $\boldsymbol{a^T}$ represents the coefficients of $y(x)$ and X is a set of variables.

2.2.4 Converting the Integral-Differential Equation to a System of Algebraic Equations

One can construct a system of algebraic equations from the integral-differential equation. The problem is transformed into a solvable system of linear equations by means of the matrices μ and η together with other transformations. The change follows here.:

$$a^T B = \gamma_0 \quad (7)$$

Here γ_0 is a vector of boundary constants and B is a matrix produced from other function values and boundary conditions. The solution y(x) of the integral-differential equation is found following application of the boundary conditions to the system of algebraic equations.

2.2.5 Converting Integrals to Numerical Form

In integral equations such as

$$\int_a^b K_1(x,t)y(t)\, dt$$

Numerical methods are used to convert the integrals into matrix formulas. For this purpose, the matrices μ and η are used in numerical computations. The resulting formula takes the following form:

$$\int_a^x K_1(x,t)y(t)\, dt = a^T \iota X_n \quad (8)$$

And for the Volterra integral part, we similarly have:

$$\int_a^b K_2(x,t)y(t)\, dt = a^T V X_n \quad (9)$$

Here, $X_n$represents the series of basis functions used to approximate the solution of the equation.

Ultimately, these theorems and formulae illustrate how the Tau approach may convert intricate integral-differential problems into a system of algebraic equations, employing straightforward matrices for numerical computations and resolving the integral equations. The application of matrices μ, η, and ι facilitates the precise resolution of integral equations..

2.3. Numerical Formulation of the Problem

The numerical framework for solving linear integrated and unwanted equations of the rope approach is presented in this section. The goal is to reduce the integrated and unwanted problem in the appropriate collection of algebraic equations. Originally, it was written numerically, the problem is then solved in numerical methods.

The general form of the integral-differential equation is as follows:

$$Ly(x) := Dy(x) + \lambda \int_a^b K_1(x,t)y(t)\,dt + \int_x^a K_2(x,t)y(t)\,dt = f(x) \quad (10)$$

Using the Tau approach, all equation components must first be stated in terms of matrices and vectors before one may solve this problem. Using polynials, the Tau approach approximates the function y(x) and converts the equation into a system of algebraic equations.

2.3.1. Converting the Integral Equation to Matrix Form

First, the integral-differential equation is transformed into the following matrix form utilizing the outcomes of the earlier phases. Correctly expressed as a vector b^T f derived from matrix multiplication, the function f(x):

$$f(x) = b^T f V^{-1} \quad (11)$$

Here:

a. $f(x)$ is a given function.

b. $\boldsymbol{b^T}$ is a vector of coefficients, used to transform the equation into an algebraic system.

c. $\boldsymbol{V^{-1}}$ is the inverse of the matrix V, which is used in solving the system of equations.

2.3.2. Algebraic System of Equations

To solve the integral-differential equation, the equation must first be converted into a system of algebraic equations. Using the basis matrices and boundary conditions, the following algebraic system is obtained:

$$a^T Bj = \gamma_j, j = 0,1,\dots,\nu - 1 \quad (12)$$

$$a^T \pi_i = f_{bi}, i = 0,1,\dots,d_f \quad (13)$$

$$a^T \pi_i = 0, i \geq d_f + 1 \quad (14)$$

In the above equations:

a. B is a matrix derived from the boundary conditions.

b. $\boldsymbol{\pi_i}$ are the vectors corresponding to the boundary conditions and other variables of the equation.

c. $\boldsymbol{f_{bi}}$ are the values on the right-hand side of the equations.

d. $\boldsymbol{\gamma_j}$ are the constants related to the boundary conditions.

e. $\boldsymbol{a^T}$ is the solution vector, which contains the coefficients of the function $y(x)$.

2.3.3. Transforming to Matrix Form for Solving the System

We employ matrices B and G to convert the system of equations into a solvable format. These matrices ultimately result in a system of linear equations structured as follows:

$$a^T BG = \gamma_b \quad (15)$$

Here:

a. B is the matrix representing the boundary conditions.

b. G is another matrix that appears in the system of equations.

c. $\boldsymbol{\gamma_b}$ is a vector containing the right-hand side values of the equations.

This change converts the integrated derivative problem effectively into a system of algebraic equations, which can later be solved by means of numerical methods. Finally, after converting the integrated difference to an algebraic system, it can be solved using various numerical techniques, such as Gossian Aboice or Matrix-based methods. The approximate solution y (x) is achieved as a coefficient vector A^t, with improvement in accuracy, number of chain entries or the degree of selected base function increases..

2.4. Numerical Examples

Example 1: Fredholm–Volterra Integro-Differential Equation
In this example, the Fredholm–Volterra integro-differential equation is given as follows:

$$-y''(x) + xy'(x) + y(x) - \int_0^1 x^2\, t^3 y(t)\, dt + \int_x^0 xy\,(t)\, dt = -\frac{1}{8}x^5 - \frac{3}{2}x^3 - 123x^2 + 5x - 1 \qquad (16)$$

With boundary conditions $\boldsymbol{y(1) = -1}$ and $\boldsymbol{y(0) = -\frac{1}{2}}$, the exact solution of the equation is given by:

$$y(x) = -\frac{1}{2}x^3 + x - 1$$

To solve this equation using the Tau method and appropriate matrices and vectors, we reach the following numerical formulation.

Assuming $n = 10$, the equation should be solved with sufficient accuracy.

In this example, various matrices are used to solve the equation. First, the matrix b is defined as follows:

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & \frac{17}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & -1 & \frac{7}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 & \frac{5}{6} & 0 & 0 & 0 & 0 & 0 \\ -20 & 0 & 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -30 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & -42 \end{pmatrix} \qquad (17)$$

Then, the matrix B is defined as follows:

$$bB = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad (18)$$

Finally, the vector γ is defined as follows:

$$b\gamma = \begin{pmatrix} -1 \\ -1 \\ 2 \\ -1 \\ 5 \\ -123 \\ 140 \\ -3 \\ 2 \\ 0 \\ -1 \\ 8 \end{pmatrix} \quad (19)$$

The coefficient vector of the unknowns, obtained from the approximate solutions of the equation, is given as follows:

$$a_n^T = [-1,1,0,-\frac{1}{2},0,0,0,0,0,0] \quad (20)$$

The equation in this vector contains the coefficients of the variable, which is used to calculate the numerical solution.

Finally, the estimated solution of the equation that follows is achieved, using the above matrix and rope method to solve the Fredhom-Valtra Integra differential equation.:

$$y_n(x) = -\frac{1}{2} + x - \frac{1}{2}x^3 \quad (21)$$

This approximated solution, obtained using the matrices and the Tau method, closely approximates the exact solution $\boldsymbol{y(x) = -\frac{1}{2}x^3 + x - 1}$. Calculation accuracy increases with the number of terms in the series or degree of base function. As a result, the example shows that the TAU method can be used effectively to solve complex integr-derivative equations and to get an accurate numerical solution for equations with different boundary conditions..

3. Nonlinear Integral Equations

3.1. Existence and Uniqueness in Nonlinear Integral Equations

In non -linear integrated equations are one of the basic problems that should be solved the existence and uniqueness of the solution. This is important because, to use numerical methods, we must first ensure that there is at least one unique solution in the equation.

Suppose the nonlinear integral equation is given in the following form:

$$y(x) = f(x) + \int_a^b K\left(x, t, y(t)\right) dt \quad (22)$$

Here, f (x) is a given function, and k (x, t, y (t)) is a nonlinear core that depends on y (t), indicating that the equation is not linear.In order to prove the existence and uniqueness of the solution for nonlinear integrated equations, the following conditions must be fulfilled:

1. **Continuity of the function $\boldsymbol{f(x)}$:** It is assumed that the function $f(x)$ is continuous and defined on the interval $[a, b]$, meaning that for all $\boldsymbol{x \in [a, b]}$, the function $f(x)$ has a well-defined value.
2. **Continuity and boundedness of $\boldsymbol{K(x, t, y(t))}$:** The kernel $K(x, t, y(t))$ must be continuous and bounded with respect to $y(t)$. In other words, the function $\boldsymbol{K(x, t, y(t))}$ must satisfy the Lipschitz condition:

$$G(x, y_1) - G(x, y_2) \mid \leq L \mid y_1 - y_2 \mid, \forall y_1, y_2 \in [\alpha, \beta] \quad (23)$$

where:

a. $G(x, y)$ is a function derived from the integral kernel $K(x, t, y(t))$.

b. α and β are constants that correspond to the bounds of the function values $y(x)$.

c. L is the Lipschitz constant, a positive fixed number that limits the rate of change of the function relative to the changes in the input.

3. **Existence and Uniqueness of the Solution:** If the above conditions are met, it follows that the nonlinear integral equation

$$y(x) = f(x) + \int_a^b K\,(x, t, y(t))dt$$

It is a constant and unique solution. In other words, if G (x, y (x)) is a feature that satisfies the Lipschitz position, it will be a unique solution to the interval [a, b] non -linear integrated equation.

Therefore, -linear integrated equations are not achieved by proving the existence and uniqueness of the solution in the Lipschitz position. This situation ensures that small changes in input function y (x) lead to small changes in solutions y (x), thus guaranteeing that the solution is unique.

3.2. Approximation of the Nonlinear Function

In this section, we focus on estimating the facility of non-lengeous facilities displayed in non-lenient integrated equations. It is believed that the function G (x, y (x)), which appears in the non -linear integrated equation, is a non -linear function that should be estimated for use in the numerical methods, such as the TAU method. To achieve this, linear function G (x, y (x)) has not been estimated as a variety of variations, enabling its effective use in numerical methods. These multiplication chains, by using appropriate base functions, do not make the linear function of a solution.This adjacent process is important to solve non-linear integrated equations, as it does not allow linear words to be handled using well-installed numerical techniques. By estimating G(x, y (x)) as a series, we can convert the equation into a form that is more manageable to calculate. The choice of base functions and the degree of polynomial chain will determine the accuracy and convergence of accuracy..

$$G\big(x, y(x)\big) \approx \sum_{i=0}^{n} \gamma_i\,(x) y_i(x) \qquad (24)$$

In this context:

a. $\gamma_i(x)$ are the coefficients dependent on x.

b. $y_i(x)$ are the approximate values of the function $y(x)$ that are obtained during the computation.

This approximation series allows us to transform the nonlinear function $G(x, y(x))$ into a set of linear functions, which can then be solved using numerical methods.

This series expansion facilitates converting the nonlinear function $G(x, y(x))$ into a collection of linear functions that are numerically solvable. Here, the following lemma is used to show that this series can be appropriately approximated:

Lemma for Nonlinear Function Approximation

Assume that the function $v(x)$ can be represented as an infinite polynomial series:

$$v(x) = \sum_{i=0}^{\infty} v_i \phi_i(x) \qquad (25)$$

In this context:

a. v_i are the coefficients of the function $v(x)$, expressed as $v^T = [v_0, v_1, v_2, \ldots\ ]$.

b. $v(x)$ represents the basis functions used for approximating the target function.

This function $v(x)$ is generally obtained in vector form as follows:

$$v(x) = v^T \mathrm{X} \qquad (26)$$

In this equation:

a. $\mathrm{X} = [1, x, x^2, \dots]$ is a vector of basis functions.

b. v^T is a vector of the function coefficients.

c. It is assumed that $p \in \mathbb{N}$, which denotes the polynomial degree.

Then, for each pp, the value of the function $v_p(x)$ is approximated as follows:

$$v_p(x) = v^T \mathrm{B}_{p-1} \mathrm{X} \tag{27}$$

In this equation:

a. B_{p-1} is a Toeplitz matrix, where each row and each column is arranged in a triangular form.

The matrix B_{p-1} is given as follows:

$$\mathrm{B} = \begin{pmatrix} v_0^T & v_1^T & v_2^T & \dots \\ 0 & v_0^T & v_1^T & \dots \\ 0 & 0 & v_0^T & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{28}$$

Variably with every p, this matrix is an upper triangular matrix.

These matrices and poisson series allow the nonlinear function G(x, y(x)) to be finally transformed into a polyn approximation fit for numerical methods including the Tau approach. This method continuously solves the nonlinear integral equation by updating the coefficients γ_i (x) and y_i (x).

3.3. Approximation Operations for Volterra-Hammerstein Integral Equations

This section presents and examines the TAU approach for Volterra-Hamstein integrated equations. The objective is to tackle the unconverted equation by numerical methods. This category of linear integrated equations is extensively utilized in scientific and engineering applications..

The equation under consideration is given as follows:

$$G\big(t, y(t)\big) \approx \sum_{p=0}^{n} \gamma_p(t) y_p(t) \tag{29}$$

In this equation:

a. $G(t, y(t))$ is the nonlinear function that needs to be approximated.

b. $\gamma_p(t)$ are the coefficients of the function, which are represented as polynomial series for each pp.

c. $y_p(t)$ are the approximate values of the function $y(t)$ that are updated throughout the computations.

Now, for applying the Tau method, the integral equation is approximated as follows:

$$y(x) = f(x) + \sum_{p=0}^{n} \int_0^x K(x,t) \gamma_p(t) y_p(t)\, dt \tag{30}$$

In this case:

a. $K(x,t)$ is the integral kernel that applies the function $y(t)$ in modeling the nonlinear integral equation.

b. $f(x)$ is the given function that appears directly in the equation.

c. $\gamma_p(t)$ are the basis functions that are applied sequentially in the equation.

This relation represents an equation that involves a series of integrals and various coefficients. Here, the equation is iteratively updated to obtain an approximate solution for $y(x)$.

3.3.1 Matrix Form of the Nonlinear Integral Equation

To represent the above equation in matrix form, we use standard notation common in integral equations. The equation can be transformed into matrix form as follows:

$$a^T\mathrm{X} = f^T\mathrm{X} + \boldsymbol{M_0}\mathrm{X} + \sum_{p=1}^{n} a^T\mathrm{B}_{p-1}M_p\mathrm{X} \quad (31)$$

In this equation:

a. $a^T\mathrm{X}$ represents the approximate solutions of the function.

b. $f^T\mathrm{X}$ represents the given values of the function $f(x)$, which appear as constants in the equation.

c. M_0 and M_p are matrices derived from the combination of coefficients and the integral kernel $K(x,t)$.

d. B_{p-1} are upper triangular matrices constructed to define the structure of the equation.

Using these relations and matrices, the nonlinear integral equation can be transformed into a system of linear algebraic equations, which can then be easily solved.

3.3.2. Formulation of the Algebraic System

At this stage, the equation is transformed into a system of algebraic equations that can be solved numerically. The system of equations is written as follows:

$$a^T\mathrm{X} = f^T\mathrm{X} + M.\mathrm{B}^{-1}\mathrm{X} + \sum_{p=1}^{n} a^T\mathrm{B}^{-1}M_p\mathrm{B}^{-1}\mathrm{X} \quad (32)$$

In this equation:

a. B^{-1} is the inverse of the matrix B, which is continuously used in the computational steps.

b. X is a vector that contains the approximate values for the solution of the equation.

By using these formulas and transforming the system into matrix form, the solution of the equation can be obtained with higher accuracy.

3.4. Construction of the Tau Approximate System

In this section, the construction of the Tau estimated system for Volterra-Hammerstein integro-differential equations has been discussed. Here, the goal is to present mathematical methods and formulas for the construction of a system of nonlinear equations when using matrix operations and special structures. In order to create an algebraic system for numerical solutions of a non-linear integrated equation, more matrices, such as M, B_(P-1), and B, must be calculated first. These Matriss Voltra-Hamstein are required for the exact solution of integrating difference equations and will eventually be used in the process of numerical solution..

3.4.1 Algebraic Form and Structure of the Nonlinear System

To simplify the solution of the equation and its implementation, the integral equation is converted into an algebraic form. It is assumed that the nonlinear part of this equation is expressed as follows:

$$a^T\mathrm{X} = f^T\mathrm{X} + a^T\mathrm{B}_{p-1}M\mathrm{X} \quad (33)$$

Here:

a. a^TX represents the approximate solution obtained.

b. f^TX represents the given function values $f(x)$ that appear constantly in the equation.

c. M is the matrix formed by the coefficients and the integral kernel $K(x,t)$.

d. B_{p-1} are matrices constructed in an upper triangular form and are used for combining and simplifying the solution of the equation.

3.4.2. Conversion to a Linear Algebraic System

In this step, the equation is converted into a linear algebraic system so that it can be solved numerically. To achieve this, equation (33) is transformed into the following form:

$$a^T \mathrm{X} = f^T \mathrm{X} + M_0 \mathrm{B}^{-1} \mathrm{X} + \sum_{p=1}^{n} a^T \mathrm{B}^{-1} M_p \mathrm{B}^{-1} \mathrm{X} \quad (34)$$

Here:

a. B^{-1} is the inverse of matrix B, which is continuously used in the computations.

b. X is a vector containing the approximate values of the equation's solutions.

With this change, the complex integrated equation is converted to a linear system with equations that can be solved directly.

3.4.3. Structure of the Lower Triangular Matrix

Next, to facilitate solving the equation, low triangular matrix is used. In this section, matrices B_ (P-1) are constructed in lower triangular form, and the kronecker product is used to represent matrices:

$$\mathrm{B} = a^T \otimes \begin{pmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 & \dots \\ 0 & \mathrm{e}_1 & \mathrm{e}_2 & \dots \\ 0 & 0 & \mathrm{e}_1 & \dots \\ & & & \end{pmatrix} \quad (35)$$

Here:

a. $\mathbf{e_i}$ are the standard vectors.

b. $\otimes$ represents the Kronecker product, which is used to construct lower triangular matrices.

3.4.4. Constructing the System of Equations for Solution

Using the constructed matrices, the equation is transformed into an algebraic system of equations for solving. In this way, to solve the Volterra-Hammerstein equation, the system of equations will take the following form:

$$\mathrm{a}^T \mathrm{B}_{p-1} M = [0, g_1(\mathrm{a}_0), g_2(\mathrm{a}_0, \mathrm{a}_1), \dots, g_n(\mathrm{a}_0, \dots, \mathrm{a}_{n-1})]^T \quad (36)$$

Here:

a. $g_i(\mathrm{a}_0, \dots, \mathrm{a}_{i-1})$ are functions that are sequentially updated and depend on the previous values of aa.

Finally, by using the above relations and the structure of the system of equations, the nonlinear system can be solved numerically. This system of equations can be applied to complex Volterra-Hammerstein equations, allowing accurate results to be obtained for nonlinear integral equations.

3.5. Numerical Examples

In this section, the Tau method is applied to solve Volterra-Hammerstein integral equations and nonlinear equations using various basis functions such as Chebyshev, Legendre, and Taylor expansions, for numerical examples. Calculations were carried out using Maple software on computers with specific configurations. For all examples, the **maximum error** between the approximate and exact solutions was computed.

Example1:Volterra-Hammerstein Integral Equation

For this example, the Volterra-Hammerstein integral equation is considered in the following form:

$$y(x) = f(x) + \int_0^x K(x,t) y^2(t)\, dt, x \in [0,1] \quad (37)$$

In this case, the function $f(x)$ and the kernel $K(x,t)$ are defined as follows:

$$f(x) = -\frac{1}{4}x^5 - \frac{2}{3}x^4 - \frac{5}{6}x^3 - x^2 + 1 \quad (38)$$

$$K(x,t) = xt + 1 \quad (39)$$

The exact solution to this equation is $y(x) = x + 1$.

For the numerical implementation of this equation, the following simple matrices are considered for the Chebyshev basis functions. In this example, $n = 5$ and $p = 2$ basis functions are used:

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix} \tag{40}$$

Matrix M is defined as:

$$M = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{41}$$

Matrix B is defined as:

$$B = \begin{bmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 & \tilde{a}_5 \\ 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 \\ 0 & 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ 0 & 0 & 0 & \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 \\ 0 & 0 & 0 & 0 & \tilde{a}_0 & \tilde{a}_1 \\ 0 & 0 & 0 & 0 & 0 & \tilde{a}_0 \end{bmatrix} \tag{42}$$

The vector $\tilde{f}$is defined as:

$$\tilde{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -5 \\ 6 \\ -2 \end{bmatrix} \tag{43}$$

Using the Tau method and solving the nonlinear system of equations step by step, specifically for this example, we obtain:

$$\tilde{a}_0 = 1, \tilde{a}_1 = \tilde{a}_2 = 0, \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0 \tag{44}$$

And the exact solution of the equation is obtained as:

$$y_n(x) = a^T X = 1 + x \tag{45}$$

Where $a^T = [1,1,0,0,0,0]$.

Finally, the Tau method was effectively used to solve nonlinear Volterra integral equations, and an accurate solution similar to the analytical solution of the equation was obtained. Based on the numerical computations, the degree of the approximate solution was sufficiently accurate and demonstrated that for complex equations, the Tau method can produce precise results.

Example 2: Nonlinear Integral Equation

In this example, the nonlinear Volterra-Fredholm integral equation is given as follows:

$$y(x) = 1 + sin^2(x) - \int_0^x 3si\,n(x-t)\,y^2(t)\,dt, x \in [0,1] \quad (46)$$

Where the exact solution of the equation is considered as $y(x) = cos\,(x)$. To solve this equation using the Tau method, the standard basis with $n = 4$ has been used. Initially, the required matrices were obtained as follows:

$$M = \begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad (47)$$

And the vector f is defined as follows:

$$\mathrm{f} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} \quad (48)$$

The results for the nonlinear equations are obtained as follows:

$$\mathrm{a} = \left[1, 0, -\frac{1}{2}, 0, \frac{1}{24}\right] \quad (49)$$

Therefore, the approximate solution will be as follows:

$$y_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \quad (50)$$

These results demonstrate that the Tau method is capable of achieving high accuracy in solving nonlinear integral equations, which is clearly improved when compared to the results from previous methods. Table 1 shows the maximum errors for different basis functions:

Table 1. Maximum Errors for Different Basis Functions.

Legendre Base	Chebyshev Base	Standard Base	N
2.52E-3	4.46E-3	6.60E-3	4
3.97E-7	6.66E-7	9.45E-7	8
1.28E-11	2.07E-11	8.41E-11	12
7.10E-16	6.50E-16	8.91E-16	16

Numerical Results

These numerical results indicate that the TAU method produces accurate results when using individual bases and reduces calculation errors, which confirms the effectiveness of the proposed numeric methods to solve non-led Vaultra-Firedhom integrated equations.

3. Conclusion

The presence and particular characteristics of the solution of two-dimensional Vaultra-Fredholm integrated equations are investigated in this work together with numerical techniques to solve and solve them with an eccentric core. Complex physical and engineering issues include heat transport, fluid mobility and biological models benefit especially from Volterra-Fredhomm-Integrated equations. Solving these equations might be difficult using conventional approaches because of eccentric core and their complicated form. Consequently, it is essential to investigate the existence and uniqueness of solutions in great depth and create numerical approaches to solve these equations. This work aimed primarily to investigate the existence and specificity of two-dimensional Voltra-Fredhom

integrated equations with monotonal core in order to solve them with the rope technique. In this framework, the characteristics and specificity of the existence of Voltera -Peace's solutions were first examined, then, using the numerical approaches, the rope technique, they were combined into a system turned into a system of algebra. These systems were then applied loosely in integrated equations to manage colonial core under appropriate matriasis.

Especially in the presence of eccentric centers, numerical results showed that the Tau technique effectively solved the challenging Volterra-Fredholm integrated equations with high accuracy. Furthermore, the findings revealed that the accuracy of the solutions much improved as we included additional conditions in the series and raised the complexity of the base functions. It emphasizes the great potential of the rope approach in solving complicated equations with single centers. Furthermore, the suggested numerical techniques can solve integrated equations with various boundary conditions rather well.

Suggestions for Future Research:

a. Development of numerical methods for equations with more complex kernels.
b. Improvement of algorithm efficiency and acceleration of convergence.
c. Examination of specific challenges in nonlinear problems with singular kernels.
d. Testing the proposed methods on real-world physical and engineering problems.

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