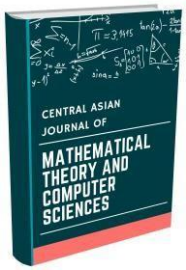

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Article

Extended Eisenstein's Criterion

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Abstract: This research paper gives an in-depth analysis of Eisenstein's criterion, as an essential tool in determining the irreducibility of polynomials over the rational numbers. It begins with exploring the theoretical foundations of the criterion and its classical conditions involving divisibility by a prime number. According to these foundations, we introduce two distinct methodological extensions for applying this criterion. The first extension, known as the direct method, investigates the suggestions of assuming that the chosen prime number divides particular non-prime components within the polynomial. The second extension, described as an indirect method, involves analyzing the polynomial's behavior under the exchange of specific prime values. Using these two methods, this paper explores the Eisenstein criterion as a domain within the bounds of possible generalizations. The results thus provide new insights into the criterion's applicability and applications, contributing to a deeper understanding of the irreducibility of many algebraic contexts.

Keywords: extended, eisenstein's, criterion, reduction, rational numbers

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1. Introduction

Eisenstein's criterion in mathematics, provides a sufficient condition for determining the irreducibility of polynomial with integer coefficient over the field of rational numbers. Especially, it provides a method to establish that a given polynomial can't be separate into the multiplying of two non-constant polynomials with rational coefficients [1], [2], [3]. Though this criterion is not universally applicable to all irreducible polynomials with integer coefficients, it confirms to be a powerful tool in many important cases, allowing irreducibility to be demonstrated by minimal effort. This criterion can often be applied either after an appropriate transformation of the polynomial or directly.

This criterion is named after the mathematician Gotthold Eisenstein [4]. Yet, it was also known as the Schönemann-Eisenstein Theorem, at the beginning of 20th century, in acknowledgment of Theodor Schönemann, who was the first to public announce the results.

One formulation of the criterion asserts that the polynomial $(x-a)^n + pF(x)$is irreducible modulo p^2, provided that $F(x)$ modulo p does not contain the factor $x-a$. This version incorporates a shift of the variable from 0 to a [5]. The condition on $F(x)$ implies that $F(a)$ is not dividable by p, and hence $pF(a)$ is dividable by p but not by p^2. However, this formulation is not entirely accurate without additional assumptions, particularly regarding the level of the polynomial $F(x)$. Without such a constraint, the level of the resulting polynomial may differ from the expected degree n, as illustrated by the counter example $x^2 + p(x^3 + 1) \equiv (x^2 + p)(px + 1) mod p^2$. This example shows that the conclusion may not hold if the degree of $F(x)$ exceeds n [6].

If we assume, however, that the degree of $F(x)$ does not surpass n, then the criterion holds and is, in fact, stronger than the original statement [7], [8], [9]. Under this assumption, if $(x-a)^n + pF(x)$ is irreducible modulo p^2, then it certainly cannot be factored into non-constant polynomials in $Z[x]$.

Subsequently, in 1850, Eisenstein published an alternative edition of the criterion in Crelle's Journal [10]. Translated into modern terms, it says: *"If a polynomial $F(x)$ of arbitrary degree has a leading coefficient equal to 1, and all subsequent coefficients are integers (real or complex) divisible by a certain prime number m, and the constant term is equal to εm varepsilon, where ε varepsilon is not divisible by m, then the polynomial $F(x)$ cannot be factored into polynomials with coefficients of the same type."* [11]

2. Materials and Methods

1) Eisenstein's Criterion: Statement and Applications

Let R be a unique factorization domain (UFD) and $p \in R$ be a prime element. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$ is a monic polynomial such that:

p divides a_i for all $0< i < n-1$, and p^2 does not divide a_0.

Then, $f(x)$ is irreducible in $R[x]$ and consequently in $F[x]$, where F is the field of fractions of R.

Example Application 1.1:

A classic illustration of this criterion is the polynomial:

$x^6 + 10x^2 + 15x + 5$

which is irreducible over Z

The prime 5 divides all coefficients except the leading term,

Yet $5^2 = 25$ does not divide the constant term 5.

However, applying Eisenstein's Criterion is not always straightforward, and certain modifications may be necessary. Below, we examine two such non-trivial cases.

2) Eisenstein's Criterion (Formal Statement)

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients. If there exists a prime p satisfying:

p divides each a_i for $0\leq i < n$,

p doesn't divide a_n, and

p^2 doesn't divide a_0 ,

then $Q(x)$ is irreducible over Q.

Additional Remarks 2.1:

- If $Q(x)$ is primitive (i.e., the greatest common divisor of its coefficients is 1), then it is also irreducible over Z.
- If $Q(x)$ is not primitive, it can be made so by dividing by its content (the GCD of its coefficients). This transformation preserves irreducibility over Q and may allow the application of Eisenstein's Criterion with a different prime p.

Key Observations 2.2 :

1. **Primitivity Requirement:**
   - For $Q(x)$ to be irreducible over Z, it must be primitive. Otherwise, it may factor into a constant and a primitive polynomial.
   - Example: $2x^2 + 4x + 6$ is reducible over Z (as $2(x^2 + 2x + 3)$),
   - but $x^2 + 2x + 3x^2 + 2x + 3$ is irreducible.
2. **Prime Independence:**
   - The prime p used in Eisenstein's Criterion must differ from any prime dividing the content of $Q(x)$

3) Advanced Perspective: Eisenstein's Criterion via Newton Polygons and Ramification Theory:

1. Newton Polygons and p-adic Valuations

Let $Q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an Eisenstein polynomial with respect to a prime p. To analyze its irreducibility using p-adic methods, we construct its Newton polygon by plotting the points:

$(0, v_p(a_0)), (1, v_p(a_1)), \ldots, (n-1, v_p(a_{n-1})), (n, 0)$,

where $v_p(a_i)$ denotes the p-adic valuation of a_i (the highest power of p dividing a_i)

Key Observations 3.1:

1) Given Conditions:

- $v_p(a_i) \geq 1$ for $0 \leq i < n$, (since $p | a_i$).
- $v_p(a_0) = 1$ (since $p^2 \nmid a_0$).

2) Lower Convex Envelope:

The Newton polygon reduces to an individual line segment from $(0,1)$ to $(n, 0)$ with slope $-1/n$.

3. Results and Discussion

Implications for Irreducibility 3.2:

- To each root α of Q has p-adic valuation $v_p(\alpha) = 1/n$.
- No proper subset of roots can have a product with integer valuation, proving Q is irreducible over Q_p.

Comparison with Classical Proof 3.3: While the standard proof via reduction modulo p is simpler, this approach reveals deeper connections to ramification theory and the geometry of p-adic fields.

1. Ramification Theory and Eisenstein Primes

The Newton polygon method clarifies when Eisenstein's Criterion applies following the change of variables [12].

Critical Constraint 3.4: Only primes dividing in the extension $Q(\alpha)$ (where α is a root of Q) can satisfy the criterion. These primes are determined by the discriminant ΔQ.

Examples 3.5:

1. Quadratic Case:

For $Q(x) = x^2 + x + 2$, $\Delta Q = -7$. Thus, only $p = 7$ might yield an Eisenstein translate.

- Modulo 7, Q becomes $(x-3)^2$, reflecting total ramification.
- A shift $x \to x + 3$ transforms Q into an Eisenstein polynomial.

2. Cyclotomic Case:

The p-th cyclotomic polynomial $Q_{p(x)}$ satisfies:

- $Q_{p(x)} \equiv (x-1)_{p-1} (mod\ p)$ Its discriminant is $\pm p - 2$, confirming p is totally ramified.

General Principle 3.6:

- Totally Ramified Primes: A prime p is **totally ramified** in $Q(\alpha)$ if and only if $\alpha\alpha$ is a basis of an Eisenstein polynomial at p.
- Quadratic vs. Higher Degrees: In quadratic extensions, result is always total (as seen above). For higher degrees, partial result may occur, but Eisenstein polynomials require total result [13].

2. Algorithmic and Theoretical Implications

1. Finding Eisenstein Translates:

To test if Q admits an Eisenstein shift for some p:

- Compute ΔQ and factorize it.
- For each prime p dividing ΔQ, check if a substitution $x \to x + c$ (for $c \in Z$) yields an Eisenstein polynomial.

2. Limitations:

The criterion is rarely applicable directly—most polynomials are not Eisenstein for any p. However, the Newton polygon method extends its utility by identifying potential ramified primes [14], [15].

Example 3.7:

$f(x) = x^5 + 9x^4 + 12x^2 + 6 \in Z[x] \subseteq Q[x]$ irreducible over Q or not ?

Solution:

Irreducible over Q by Eisensteins with $p = 3$

Sum $3 \nmid 1, 3/9, 3/12, 3/6$ and $3^2 = 9 \nmid 6$

Example 3.8 :

The polynomial $3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible over Q because $5 \nmid 3$ and $25 \nmid 20$ but 5 does divide $15, -20, 10$ and 20.

Example 3.9:

$P(x) = x^4 + 3x^3 - 9x^2 + 18x - 12 \in Z[x] \subseteq Q[x]$ irreducible over Q or not ?

Solution:

$P = 3, 3/12$, but $p^2 = 9 \nmid 12$ then p is irreducible over Q.

Example 3.10:

Use the Eisenstein criterion to show the following polynomials are irreducible over Q.

a) 426-b) $x^4 + 142x^3$ b) $x^3 - 26x + 13$

Solution:

a) $p = 13$ divides $13, -26, 13 \nmid 1, 13^2 \nmid 13$

$\therefore$ by Eisenstein criterion $\Rightarrow x^3 - 26x + 13$ is irreducible over Q.

b) $p = 2, 2/-426, 142, 0, 2 \nmid 1, 2^2 \nmid -426$

$\therefore$ by Eisenstein criterion $\Rightarrow x^4 + 142x^3 - 426$ is irreducible over Q.

Example 3.11:

Prove that $f(x) = 4x^5 - 7x^3 + 21x^2 + 28$ is irreducible over Q.

Proof :

Note $p = 7$ is prime and $p/28, p/21, p/-7$ but $p \nmid 4$ and $p^2 = 49 \nmid 28$ this means f is irreducible over Q by Eisenstein criterion.

4. Conclusion

The research confirms that Khorezm bakhshis played a pivotal role in shaping and transmitting the epic tradition of Uzbek oral literature. Through figures such as Oshiq Oydin Pir, this tradition merged spiritual, artistic, and mythological dimensions, resulting in a complex oral culture that influenced both regional and transnational epic narratives. The connection of these performers to ancient Zoroastrian and Islamic Sufi traditions illustrates the continuity and adaptability of cultural memory in Central Asia. Furthermore, the symbolic structure of dastans and their embedded values of heroism, devotion, and mentorship demonstrate the educational and ideological functions of the bakhshi in society. This study contributes to a deeper understanding of how folklore, performance art, and spiritual mentorship coalesced in Khorezm to create a dynamic and enduring literary legacy. Future research may explore comparative studies with other Turkic epic traditions or delve into the musical and performative aspects of bakhshi artistry for a fuller appreciation of their cultural role.

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