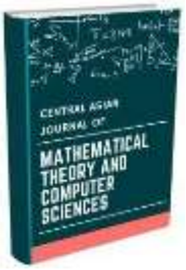

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Article

Some Results on Rough k-Space

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Abstract: This paper introduces the concept of rough k-space within the framework of rough set theory. The primary aim of this work is to define rough k-space and explore its properties, including rough continuity, homeomorphisms, and topological characteristics. Specifically, it is shown that the restriction of a rough continuous function to any rough compact subset of a rough space remains rough continuous. Additionally, the cross product of a rough k-space with a rough compact T_2-space results in a rough k-space. The study also highlights key hereditary and topological properties of rough k-spaces. The novelty of this research lies in its extension of rough set theory to include the concept of rough k-spaces, which integrates topological and rough set properties, and introduces a new approach to understanding the interaction between rough sets and continuous functions. Furthermore, the paper provides detailed results on the continuity and homeomorphism properties of rough k-spaces, offering a fresh perspective on their application in mathematical and computational contexts. The implications of these findings are significant for further research in rough topology, particularly in the development of robust mathematical models for rough set theory and its applications in areas such as decision-making, knowledge discovery, and artificial intelligence.

Keywords: Hausdorff property; Kelley-space; rough set; rough topological space; rough Kelley space; rough open map; rough closed map

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1. Introduction

Some drawbacks are there in category of CW complexes. A well-known fact is that product of two CW complexes spaces not necessary be a CW complex [1], [2]. A suitable condition for proving this case presented by Whitehead and it is written ***k-space***. He provided a Hausdorff space as a necessary condition in his definition. Subsequently, in 1954, Cohen utilized Whitehead's definition to establish that the Cartesian product of two spaces constitutes a k-space if one is locally compact Hausdorff and the other is a k-space [3]. In 1955, Kelley as an alternative defined k-space without referencing Hausdorff spaces, presenting the final topology for the embeddings of its compact closed subspaces. Kelley space was another name for such spaces. However, the Hausdorff principle was applied in another formula to guarantee the closedness of compact subsets. [4], [5]. Zdzisław Pawlak, a Polish computer scientist, developed a novel idea in 1982 to categorize objects and imprecise information addressing as rough set theory. It met the upper and lower approximations of the original set, two exact concepts that were the same as its approximations. The imperfection is recognised by boundary region. When its boundary region is null, this leads to the set is ***exact*** or ***crisp***, otherwise it is ***rough***. Furthermore, the interior of the set represented by the lower approximation and the upper approximation represents the closure, this set depending on an equivalence relation [6]. For this, rough spaces, which provided lower and upper approximations of any subset, were demonstrated by Thivagar et al. In fact, a lot of cases solved using these spaces. Also, in the same year, in 2012, Mathew and John satisfied numerous rough space conditions by using a novel topological technique for rough sets [7]. Four years later, Ravindran and Dhivya introduced a fresh study that took into account compactness and connectivity in rough spaces by incorporating Mathew's method into John's. They also worked out the rough separation terms T_1 and T_2 [8].

During this paper, a new concept is presented in rough theory which is rough k-space. Our goal is to present a few of this term's characteristics and how a rough homeomorphism effects on such spaces. Overall, the second section handles the basic principles like definition of k- space, roughness of a set, rough topology and few related properties. The third section contains major outcomes regarding the concepts of rough k-space, rough continuity, rough homeomorphism, and cross product.

2. Materials and Methods

PRELIMINARIES

2.1 **Definition:** A Hausdorff topological space (X, τ) is called ***k-space*** provided that for each subset F of X satisfying $F \cap C$ is closed in C (equivalently, in X) for every compact subset C of X, then F is itself closed.

2.2 **Proposition:** If f is a map from a k-space (X, τ) into a topological space (Y, γ) then f is ***continuous (cont.)*** if for any compact set $C \subseteq X$, $f \mid C$ (restriction of f by C) is continuous.

2.3 **Proposition:** The Cartesian product $(X \times Y)$ of a k-space (X, τ) and a compact Hausdorff space (Y, γ), is also a k- space.

Suppose we have a non empty universe U, **and an equivalence relation R on U is said to be an *indiscernibility relation*. Then the order pair (U, R) is said to be an** *approximation space* **[1]. For a subset X of U we need to** specify **the set X due to the relation R, we associate two exact sets bellow called** *upper* **and** *lower approximations* **supposing R_x represents the equivalence class related to x.**

2.4 **Definition [3]: In an approximation space (U, R) if X is any set in U then:**

(1) $R_*(X) = \cup\{ x \in U: R_x \subseteq X \}$ **is said to be the *lower approximation (l. approx.)* of X.**

(2) $R^*(X) = \cup\{ x \in U: R_x \cap X \neq \emptyset \}$ **is said to be the *upper approximation (u. approx.)* of X.**

(3) $BN_R(X) = R^*(X) - R_*(X)$ is said to be **the *boundary* of** X.

(4) A set X is said to be ***rough set (r- set)*** if the boundary region of X is not null, otherwise it is ***exact.***

2.5 **Example:** Consider, $U = \{a, b, c, d\}$ and

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a)\}.$$

If $X = \{a, d\}$, then we have $U/R = \{\{a, c\}, \{b\}, \{d\}\}$.

We have $R_*(X) = \{d\}$, $R^*(X) = \{a, c, d\}$ and $BN_R(X) = \{a, c\}$.

Since $R^*(X), R_*(X)$ are not equal therefore X is a r- set in (U, R).

2.6. Example: Let $U = (0, \infty)$ and R is an equivalence relation U defined as below:

$$U/_R = \{(0,1], (1,2], (2,3], \dots, (k, k+1], \dots\}$$

where $(k, k+1]$, $k = 0,1,2, \dots$ stand for semi-open intervals. Then its approx. space is (U, R). Consider approximations of open interval $X = (0, r)$, when $n < r < n + 1$ for an integer $n \geq 0$. Then:

$R_*(X) = \cup_{i=0}^{n-1}(i, i+1] = (0, n]$

$R^*(X) = \cup_{i=0}^{n} (i, i+1] = (0, n+1]$

This means that, X is a r-set in its approx. space (U, R).

From now on, we will use notation $R(X) = (R_*(X), R^*(X))$ to refer a r- set.

2.7. Definition: Suppose $R(X) = (R_*(X), R^*(X))$ be a r-set in an approx. space (U, R), and τ_*, τ^* are topologies consist of just exact sets in $R_*(X)$ and $R^*(X)$ respectively. Then, the set $\tau = (\tau_*, \tau^*)$ is said to be a ***rough (r-) topology*** on $R(X)$ and $(R(X), \tau)$ is ***rough(r-) topological space***. In a r-topology $\tau = (\tau_*, \tau^*)$, τ_* is known as ***lower(l.) r-topology***, while τ^* is ***upper(u.) r-topology***.
2.8. Definitio: assume that, $G = (G_*, G^*)$ is a r-set in a r-topological space $(R(X), \tau)$. Then the set G_* is said to be ***lower rough (l. r-) open*** when $G_* \subseteq \tau_*$. Also, G^* is said to be ***upper rough (u. r-) open*** if $G^* \subseteq \tau^*$. The r-subset G is said to be ***rough(r-) open*** if G is l. r-open and u. r-open.
2.9. Example: Suppose $R(X) = (R_*(X), R^*(X))$ is a r-set and $\tau_* = \{A \subseteq R_*(X) \mid A \text{ is exact}\}$ and $\tau^* = \{B \subseteq R^*(X) \mid B \text{ is exact}\}$. Therefore, each of τ_* and τ^* represent topologies on $R_*(X)$ and $R^*(X)$ respectively. Then, the r- topology $\tau = (\tau_*, \tau^*)$ on $R(X)$ is said to be ***discrete r- topology*** on $R(X)$.
2.10. Example: Consider $R(X) = \left(R_*(X), R^*(X)\right)$ as a r-set and suppose $\tau_* = \{\phi, R_*(X)\}$ while $\tau^* = \{\phi, R^*(X)\}$. It straightforward τ_* and τ^* represent topologies on $R_*(X)$ and $R^*(X)$. The r- topology $\tau = (\tau_*, \tau^*)$ is named ***indiscrete r-topology*** on $R(X)$.
2.11. Definition: Let $(R(X), \tau)$ be a r- topological space ,then $N_* \subseteq R_*(X)$, named a τ_*-*neighbourhood of* $x \in R(X)$ if we have a l. r- open set A_1 in $R_*(X)$ satisfies, $x \in A_1 \subseteq N_*$. Similarly, the set $N^* \subseteq R^*(X)$, named a τ^*- *neighbourhood of* x if we have an u. r-open set A_2 in $R^*(X)$ satisfies, $x \in A_2 \subseteq N^*$. When $N_* \subseteq R_*(X) \subseteq N^* \subseteq R^*(X)$ then, $N = (N_*, N^*)$ named *r- neighbourhood of x*.
2.12. Definition : Let $(R(X), \tau)$**,** be a r-topological space, where $R(X) = \left(R_*(X), R^*(X)\right)$ and $\tau = (\tau_*, \tau^*)$. If τ_* is Hausdorff, then it is named ***lower (l.) Hausdorff topology*** and if τ^* is a Hausdorff, then it is named ***upper (u.) Hausdorff topology.*** If each of τ_* and τ^* is Hausdorff, then $(R(X), \tau)$ **is named a *r-Hausdorff topological space.***
2.13. Definition: A subset $A = (A_*, A^*)$ of a r-topological space $(R(X), \tau)$ **is named *l. r-closed* if** ${A_*}^C = R_*(X) - A_*$ is l. r-open. Also A is said to be ***u. r- closed*** if ${A^*}^C = R^*(X) - A^*$ is u. r-open. The set $A = (A_*, A^*)$ is called ***r-closed*** if A is l. r-closed and u. r-closed.
2.14. Definition: Suppose $(R(X), \tau)$ and $(R(Y), \gamma)$ are r-topological spaces with topologies $\tau = (\tau_*, \tau^*)$ and $\gamma = (\gamma_*, \gamma^*)$, respectively. If $f_*: (R_*(X), \tau_*) \longrightarrow (R_*(Y), \gamma_*)$ is cont. map at $x \in X$ then it is called ***lower continuous(l. cont.)*** *at* x. If $f^*: (R^*(X), \tau^*) \longrightarrow (R^*(Y), \gamma^*)$ is cont. map at $x \in X$ then it is called ***upper continuous(u. cont.)*** *at* x. Then, $f = (f_*, f^*): (R(X), \tau) \longrightarrow (R(Y), \gamma)$ called ***rough continuous (r- cont.) map at x*** when f_* and f^* are l. cont. and u. cont. at x respectively.
We say $f : (R(X), \tau) \longrightarrow (R(Y), \gamma)$ is ***rough continuous (r- cont.)*** when f is r- cont. at each x in X.
2.15. Definition: Let $R(X) = (R_*(X), R^*(X))$ be a r-set. If $R_*(X)$ is a compact set then $R_*(X)$ called a ***lower compact (l. comp.) set.*** Similarly, if $R^*(X)$ is a compact set then $R^*(X)$ is called ***upper compact (u. comp.) set.*** Then $R(X) = (R_*(X), R^*(X))$ is named a ***rough compact (r- comp.)*** and if $R_*(X)$ and $R^*(X)$ are l. comp. and u. comp. respectively.

3. Results and Discussion

This section motivates for introducing a new concept in rough theory called *rough k- space*. In addition, new related properties will be studied. Initially, rough k- space will be defined as follows:
3.1. Definition: Assume $(R(X), \tau)$ be a r- Hausdorff space. Then $(R_*(X), \tau_*)$ is named ***lower(l.) k-space*** if for any set W_* in $R_*(X)$ **satisfying** $W_* \cap K_*$ **is l. closed in** K_* **(equivalently, in** $R_*(X)$**) for every l. comp. set** K_* of $R_*(X)$ **then** W_* **is l. closed. Also,** $(R^*(X), \tau^*)$ named ***upper(u.) k- space*** if for each set W^* in $R^*(X)$ **satisfying** $W^* \cap K^*$ is **u. closed in** K^* **(equivalently, in** $R^*(X)$**) for every u. comp. set** K^* of $R^*(X)$ **then** W^* **is u. closed.** The space $(R(X), \tau)$ is called ***rough(r-) k-space*** if $R_*(X)$ is a l. k-space and $R^*(X)$ is an u. k-space [9].

3.2. Theorem: If $(\boldsymbol{R}(\boldsymbol{X}), \boldsymbol{\tau})$ is r-$\boldsymbol{k}$-space and $(\boldsymbol{R}(Y), \boldsymbol{\gamma})$ is any r-topological space, then $\boldsymbol{f}: (\boldsymbol{R}(\boldsymbol{X}), \boldsymbol{\tau}) \longrightarrow (\boldsymbol{R}(Y), \boldsymbol{\gamma})$ is r- cont. function if and only if for any r-comp. set $\boldsymbol{C} = (\boldsymbol{C}_*, \boldsymbol{C}^*)$ of $\boldsymbol{R}(\boldsymbol{X})$, the restriction $\boldsymbol{f} \,|\boldsymbol{C}$ is r-cont. function [10], [11], [12].

Proof: suppose f is a r-cont. function defined on the r- k-space $(R(X), \tau)$ into r- space $(R(Y), \gamma)$. Then, $f_*: (R_*(X), \tau_*) \longrightarrow (R_*(Y), \gamma_*)$ is l. cont. and $f^*: (R^*(X), \tau^*) \longrightarrow (R^*(Y), \gamma^*)$ is u. cont.. Since $(R_*(X), \tau_*)$ and $(R^*(X), \tau^*)$ are l. k-space and u. k-space respectively and depending on proposition (2.2), for any r-comp. set $C = (C_*, C^*)$ of $R(X)$, the restrictions $f_*|C_*$ is l. cont. and $f^*|C^*$ is u. cont.. Therefore, $f|C = (f_*|C_*, f^*|C^*)$ is r-cont..

Conversely, let $f\,|C$ r-cont. function for every r-comp. subset $C = (C_*, C^*)$ of $(R(X), \tau)$. This means, the restrictions $f_*|C_*$ and $f^*|C^*$ are l. cont. and u. cont. functions respectively where C_* and C^* are l. comp. and u. comp. sets in $(R_*(X), \tau_*)$ and $(R^*(X), \tau^*)$ respectively. Now proposition (2.2) shows that $f_*: (R_*(X), \tau_*) \longrightarrow (R_*(Y), \gamma_*)$ is l. cont. and $f^*: (R^*(X), \tau^*) \longrightarrow (R^*(Y), \gamma^*)$ is u. cont. [13].

3.3. Lemma: If $K = (K_*, K^*)$ is r-comp. set and $B = (B_*, B^*)$ r- closed in $(R(X), \tau)$, then $B \cap K$ is r-comp..

Proof: If K is r-comp., then K_* and K^* are comp. in $R_*(X)$ and $R^*(X)$ respectively. If $\{V_{*i}: i \in I\}$ be an open cover for $B_* \cap K_*$, then $\{V_{*i}: i \in I\} \cup {B_*}^c$ is open cover of K_*. But K_* is comp. set, we can find subcover $\{V_{*1}, \dots, V_{*n}\} \cup {B_*}^c \supset K_*$. So, it is clear that $\{V_{*1}, \dots, V_{*n}\}$ covers $B_* \cap K_*$, so it is comp. in $R_*(X)$. Similarly, $B^* \cap K^*$ can be proved as a comp. subset of $R^*(X)$. Then $B \cap K$ is r-comp. in $R(X)$.

3.4. Proposition: A r-closed subspace of a r-k-space is a r-k-space.

Proof: Let $(R(X), \tau)$ be a r- k- space, where $R(X) = (R_*(X), R^*(X))$, $\tau = (\tau_*, \tau^*)$ and let $B = (B_*, B^*)$ be a r- closed set in $R(X)$. Let $A = (A_*, A^*)$ be subset of B. Assume for any comp. subset $K = (K_*, K^*)$ of B such that each of $A_* \cap K_*$ and $A^* \cap K^*$ are closed subsets of K_* and K^* respectively. Now let $K_* = C_* \cap B_*$ for a l. comp. set $C_* \subseteq R_*(X)$, then K_* is also l. comp. in $R_*(X)$ [Lemma 3.3]. According to the assumption, we have $C_* \cap A_* = B_* \cap A_*$ is closed subset of B_* and then closed in $R_*(X)$ (B_* is closed in $R_*(X)$) Since $R_*(X)$ is l. k-space, A_* is l. closed in $R_*(X)$, and then l. closed in B_* (B_* is closed in $R_*(X)$). Similarly, we can prove A^* is u. closed in $R^*(X)$. This means $B = (B_*, B^*)$ is r-k-space [14].

3.5. Proposition: If $(R(X), \tau)$ is a r- k- space and $(R(Y), \gamma)$ is a r- comp. and r-Hausdorff space, then $(R(X) \times R(Y), \tau \times \gamma)$ is a r-k-space.

Proof: Let $(R(X), \tau)$ r- k- space, then $(R_*(X), \tau_*)$ and $(R^*(X), \tau^*)$ be l. k- space and u. k-space respectively. Since $(R(Y), \gamma)$ r-comp. and r- Hausdorff, then $(R_*(Y), \gamma_*)$ is a l. comp. and l. Hausdorff space and $(R^*(Y), \gamma^*)$ is an u. comp. and u. Hausdorff space. So, proposition (2.3) asserts that, $R_*(X) \times R_*(Y)$ will be a l. k- space and $R^*(X) \times R^*(Y)$ will be an u. k- space, thus $R(X) \times R(Y)$ is r- k- space [15].

3.6. Definition: Let $(R(X), \tau)$ and $(R(Y), \gamma)$ are two r-spaces with topologies $\tau = (\tau_*, \tau^*)$, $\gamma = (\gamma_*, \gamma^*)$. Assume that, $f = (f_*, f^*): (R(X), \tau) \longrightarrow (R(Y), \gamma)$ is a function where $f_*: (R_*(X), \tau_*) \longrightarrow (R_*(Y), \gamma_*)$ and $f^*: (R^*(X), \tau^*) \longrightarrow (R^*(Y), \gamma^*)$. Then, $f = (f_*, f^*)$ is named ***rough(r-) homeomorphism*** when:

(1) f_* and f^* are bijective.

(2) f_* and f^* are l. cont. and u. cont. respectively.

(3) f_* and f^* are l. closed and u. closed respectively.

Also, f_* and f^* are said to be ***lower(l.) homeomorphism*** and ***upper(u.) homeomorphism*** respectively.

3.7. Theorem: Let $(R(X), \tau)$ and $(R(Y), \gamma)$ are r-Hausdorff topological spaces with topologies $\tau = (\tau_*, \tau^*)$ and $\gamma = (\gamma_*, \gamma^*)$ and let $f = (f_*, f^*): (R(X), \tau) \longrightarrow (R(Y), \gamma)$ is a r-homeomorphism. Then, $R(X)$ is r- k- space if and only if, $R(Y)$ is a r- k- space.

Proof: Firstly, consider $(R(X), \tau)$ is a r- k- space. Let $W = (W_*, W^*)$ be a r-set in $R(Y)$ and $K = (K_*, K^*)$ be a r-comp. set in $R(Y)$ for which $W \cap K = (W_* \cap K_*, W^* \cap K^*)$ is r-closed subset of K. Since $f: (R(X), \tau) \longrightarrow (R(Y), \gamma)$ is a r-homeomorphism, so the maps $f_*: (R_*(X), \tau_*) \longrightarrow (R_*(Y), \gamma_*)$ and $f^*: (R^*(X), \tau^*) \longrightarrow (R^*(Y), \gamma^*)$ are l. homeomorphism and u. homeomorphism respectively. Hence, ${f_*}^{-1}(W_*)$ is a set in $R_*(X)$, ${f_*}^{-1}(W_* \cap K_*) =$

$f_*^{-1}(W_*) \cap f_*^{-1}(K_*)$ **is l. closed and** $f_*^{-1}(K_*)$ **is l. comp. set** of $R_*(X)$ **then** $f_*^{-1}(W_*)$ **is l. closed (** $(R_*(X), \tau_*)$ is l. r-k-space) and hence, $f_*\left(f_*^{-1}(W_*)\right) = W_*$ **is l. closed in** $R_*(Y)$ (f_* **is** l. homeomorphism **). In same way we can prove,** $f^*\left(f^{*-1}(W^*)\right) = W^*$ is u. closed in $R^*(Y)$**.** Hence, W is a r-closed subset of $R(Y)$ that asserts $(R(Y), \gamma)$ is a r-k- space.
Secondly, the other part can be proved by same routine.

4. Conclusion

In this paper, a new concept, the rough k-space, was introduced in rough set theory. The paper explored several important properties of rough k-spaces, including the concept of rough continuity and rough homeomorphisms. It was shown that the rough continuous functions on a rough k-space are continuous on its compact subsets, and the Cartesian product of a rough k-space with a rough compact T2-space also forms a rough k-space. Moreover, the paper discussed how rough k-spaces retain hereditary and topological properties. These findings contribute to the expanding understanding of rough topological spaces and provide a solid foundation for further research in this area.

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