CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES
http://cajmtcs.centralasianstudies.org/index.php/CAJMTCS
Volume: 02 Issue: 06 | Jun 2021 ISSN: 2660-5309

Stability of the Galerkin Method for one Quasilinear Parabolic Type Problem

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Abstract

The article considers a parabolic-type boundary value problem with a divergent principal part, when the boundary condition contains the time derivative of the required function. An approximate solution is constructed and the stability of the Galerkin method of the problem under consideration is established. In this paper, a generalized solution of the problem under consideration is defined in the space $\widetilde{H^{1,1}}(Q_T)$. The proposed boundary value problem is considered under certain conditions for the function involved in the equation and the boundary condition, which allow the existence and uniqueness of the generalized solution. For the numerical solution of the problem under consideration, an approximate solution was constructed using the Bubnov-Galerkin method. The concept of stability of the Galerkin process for this problem is introduced. The aim of the research is to obtain a condition for the stability of the computational process of the considered mixed problem.Using the Bubnov-Galerkin method, the problem under consideration is reduced to solving a system of ordinary differential equations. Further, we consider the "perturbed" problem for the system of the Bubnov-Galerkin method and obtain estimates for the difference between the solutions of the original and perturbed systems. The article establishes the stability of the Galerkin method of the problem under consideration, under the conditions of strong minimality of the coordinate system, which allows the calculation of an approximate solution of the problem under consideration by the proposed Bubnov-Galerkin method.

A R T I C L E I N F O

Article history:
Received 29April2021
Revised form 19 May 2021
Accepted 20 Jun 2021

Keywords: Mixed problems, quasilinear equation, boundary condition, Galerkin method, generalized solution, parabolic type, approximate solution, error estimate. a priori estimates, coordinate system, monotonicity, stability, strongly minimality, inequalities, boundary, domain, dot product, continuity, error

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Introduction

When studying a number of applied problems, it becomes necessary to study boundary value problems of parabolic type, when the boundary condition contains the time derivative of the desired function. Problems of this type arise, for example, when the surface of a body, the temperature of which is the same at all its points, is washed off by a well-mixed liquid. Some nonlinear problems of parabolic type with a boundary condition containing the time derivative of the desired function were considered, for example, in works[1-3]. Many scientists were engaged in the construction of an approximate solution by the Galerkin method and obtaining a priori estimates for an approximate solution for parabolic quasilinear problems without a time derivative in the boundary condition: Mikhlin S.G., Douglas J. Jr, Dupont T., Dench J. E., Jr, Jutchell L. and others.[4-9]. And quasilinear problems, when the boundary condition contains the time derivative of the desired function using the Galerkin method, have been little studied [10-13].

Formulation of the problem In this paper, we consider a quasilinear problem of parabolic type, when the boundary condition contains the time derivative of the required function:

$$\begin{cases} u_t - \frac{d}{dx_i} a_i(x,t,u,\nabla u) + a(x,t,u,\nabla u) = 0\,, \\ a_0 u_t + a_i(x,t,u,\nabla u)\cos(\nu, x_i) = g(x,t,u), (x,t) \in S_t\,, \\ u(x,0) = u_0(x)\,, x \in \Omega \end{cases} \quad (1)$$

where Ω *is a bounded domain in* $E_m, m = \dim -$ dimension field Ω, $Q_T = \{\Omega \times [0,T]\}$, $S_T = \{\partial\Omega \times [0,T]\}$, $= const > 0$

Let's denote by $\widetilde{H^{1,1}}(Q_T) = \{u \in H^{1,1}(Q_T): a_0 u_t \in L_2(S_T)\}$.

Definition. u is called a Generalized solution of the problem (1) when $u \in \widetilde{H^{1,1}}(Q_T)$ and satisfies the following identity

$$\int_{Q_T} (u_t \eta + a_i(x,t,u,\nabla u)\eta_{x_i} + a(x,t,u,\nabla u)\eta)\, dxdt + + \int_{S_T} (a_0 u_t + g(x,t,u)))\eta)\, dxdt = 0 \quad (2)$$

$\forall\, \eta \in H^{1,1}(Q_T)$

Main results: Consider problem (1) under the following conditions

1) for $(x,t) \in \bar{Q}_T$ and arbitrary $u, v,\ p и q$ the inequalities are true

2) $|a_i(x,t,u,p) - a_i(x,t,u,q)| \le \mu |p-q|$,

$|a_i(x,t,u,p) - a_i(x,t,v,p)| \le L_0(|u|^\propto + |v|^\propto)|u-v|$,

$|a(x,t,u,p) - a(x,t,u,q)| \le L_1\big(|u|^\propto |p-q| + \big(|p|^\beta + |q|^\beta\big)|p-q|\big)$, **(3)**

$|a(x,t,u,p) - a(x,t,v,p)| \le L_2\big(|p|^{2\beta} + |u|^{2\alpha} + |v|^{2\alpha})|u-v|\big)$,

$|g(x,t,u) - g(x,t,v)| \le L_3(|u|^\gamma + |v|^\gamma)|u-v|)$,

где

$$\propto \in \begin{cases} [0;\infty), m = 2 \\ \left[0; \frac{2m}{(m-2)l}\right], m \ge 3 \end{cases}, l > m = \dim(\Omega) \quad \mathbf{(4)}$$

$$\gamma \in \begin{cases} [0;\infty), m = 2 \\ \left[0; \frac{2(m-1)}{(m-2)(l-1)}\right], m \ge 3 \end{cases}, \beta \le \frac{2}{l}$$

$\mu, L_0, L_1, L_2, L_3\ -$ *positive constants.*

2) The boundary S of the domain Ω is such that the inequalities are true[14-15]

$$\|u\|_{L\tilde{q}(\Omega)} \le \varepsilon \|\nabla u\|^2_{L_2(\Omega)} + C_\varepsilon \|u\|^2_{L_2(\Omega)},$$

$\tilde{q} = \frac{2l}{l-2}$, $l > m$; (5)

$$\|u\|_{L\bar{q}(S)} \le \varepsilon \|\nabla u\|^2_{L_2(\Omega)} + C_\varepsilon \|u\|^2_{L_2(\Omega)} ,$$

$\bar{q} < \frac{2(m-1)}{m-2}$

3) Monotonicity condition. For any functions $u, v \leq H^1$

$(a_i(x,t,u,\nabla u) - a_i(x,t,v,\nabla v), u_{x_i} - v_{x_i})_\Omega + (a(x,t,u,\nabla u) - a(x,t,v,\nabla v), u - v)_\Omega 0$ (6)

4) By$(x,t,u) \in \{\bar{\Omega} \times [o,T] \times E_1\}$function $g(x,t,u)$measurable

by (x, t, u), continuous by (t, u) and satisfies the following inequality:

$|g(x,t,u) - g(x,t,v)| \leq g_0|u - v|, g(x,t,0) \in L_2(S_T)$ (7)

Let us construct an approximate Galerkin solution. Take a coordinate system from the space $H^1(\Omega)$. We will seek an approximate solution U (x, t) in the form[16-17]

$U(x,t) = \sum_{k=1}^{n} C_k^n(t)\varphi_k(x)$(8)

where$C_k^n(t)$are determined from the system of ordinary differential equations

$(U_t, \varphi_j)_{\hat{L}_2} + (a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega + (a(x,t,U,\nabla U), \varphi_j)_\Omega =$

$= (g(x,t,U), \varphi_j)_S, j = \overline{1,n}$ (9)

with initial conditions

$(U(x,O) - u_0, \varphi_j)_{H^1(\Omega)} = 0$

Here$\hat{L}_2(\Omega)$– spaceoffunctionwith scalar product

$(u,v)_{\hat{L}_2} = (u,v)_\Omega + a_0(u,v)_s$

Suppose that the coordinate system {φ_k} ⊂〖H〗 ^ 1 (Ω) is strongly minimal in the space 〖L ̂〗 _2 (Ω) that is, there exists a constant q, does not depend on n, such that 0≤q≤ 〖q_i〗 ^ n, where 〖q_i〗 ^ n are the eigenvalues of the matrix

$Q_n = \left\{(\varphi_k, \varphi_j)_{\hat{L}_2}\right\}_{k,j=1}^{n}$

Let's write system (7) in the vector form

$$\begin{cases} Q_n \dot{C}_n(t) + R_n(t, C_n(t)) = f_n(t, C_n(t)) + g_n(t, C_n(t)) \\ C_n(t)|_{t=0} = C_n(0) \end{cases} \quad (10)$$

here

$R_n(t, C_n) = \left\{(a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega \right\}_{k=1}^{n},$

$f_n(t, C_n) = \{(a(x,t,U,\nabla U), \varphi_k)_\Omega\}_{k=1}^{n},$

$g_n(t, C_n) = \{(g(x,t,U), \varphi_k)_S\}_{k=1}^{n}, C_n = \{C_n^k\}_{k=1}^{n}$

n-dimensional vectors.

Suppose that instead of the Galerkin system (8) let's solve the "perturbed" system[18-20]

$$\begin{cases}(Q_n+\Gamma_n)\dot{\widetilde{C_n}}(t)+R_n\left(t,\tilde{C}_n(t)\right)=g_n\left(t,\tilde{C}_n(t)\right)+f_n\left(t,\tilde{C}_n(t)\right)+\delta_n(t,C_n),\\ \delta_n(t,C_n)=\sum_{i=1}^{m+1}\delta_n^i(t,C_n) \qquad (11)\\ \tilde{C}_n(t)\big|_{t=0}=\tilde{C}_n(0)=C_n(0)+\varepsilon_n\end{cases}$$

if there exist such positive constants independent of n

where C ̃_n (t) is solution of the perturbed problem.

Definition. The Galerkin process for (8) is called stable,if there exist such positive constants independent of np_i $(i=\overline{0,3})$,which for sufficiently small norms of matrices$\|\Gamma_n^0\|\|\Gamma_n\|$andnorm of vectors$\|\delta_n(t,C_n)\|_{L_2(0,t,E_n)}$, $\|\Delta_n\|_{E_n}$

holds inequality

$$\left\|\tilde{C}_n(t)-C_n(t)\right\|_{E_n}\le p_0\|\Delta_n\|_{E_n}+p_1\left\|\Gamma_n^0\right\|+p_2\left\|\Gamma_n\right\|+$$

$$+p_3\max_{\|C_n\|\le K}\|\delta_n(t,C_n)\|_{L_2(0,t,E_n)} \ (12)$$

An approximate solution U (x, t) is called stable in the space 〖L〗 _2 (Ω) if an equality similar to (12) holds for the difference

$\left\|\widetilde{U}(x,t)-U(x,t)\right\|_{L_2}$, where$\widetilde{U}(x,t)=\sum_{k=1}^{n}\tilde{C}_k^n(t)\varphi_k(x)$.

Using the strong minimality of the coordinate system$\{\varphi_k\}$in$\hat{L}_2(\Omega)$we obtain the inequalities

$$\|C_n(t)\|_{E_n}^2\le\frac{1}{q}\|U\|_{\hat{L}_2}^2\le {}^{N}\!/_{q} \ (13)$$

$$\int_0^t\left\|\frac{dC_n(t)}{dt}\right\|_{E_n}^2dt\le\frac{1}{q}\left\|\frac{dU}{dt}\right\|_{L_2(0,t,\hat{L}_2)}^2\le {}^{N}\!/_{q}$$

Then, in inequality (13), we put$K={}^{N}\!/_{q}$.

LetadmittederrorsΓ_n, $\delta_n(t,C_n)$are such that

$\|\Gamma_n\|\le\alpha q$, (14)

in the sphere $\|C_n(t)\|_{E_n}^2\le K$inequality is true

$\|\delta_n(t,C_n)\|_{L_2(0,t,E_n)}^2\le\delta C_\kappa$ (15)

We denote$z_n=\tilde{C}_n(t)-C_n(t)$.

Let us subtract (10) from (11) and the resulting equation is multiplied scalarly by the vector$z_n(t)$

$$\frac{1}{2}\frac{d}{dt}\big((Q_n+\Gamma_n)z_n,z_n\big)_{E_n}+\big(R_n(t,\tilde{C}_n)-R_n(t,C_n),z_n\big)_{E_n}+$$

$$+\big(f_n(t,\tilde{C}_n)-f_n(t,C_n),z_n\big)_{E_n}=\big(g_n(t,\tilde{C}_n)-g_n(t,C_n),z_n\big)_{E_n}+(\Phi_n(t,C_n),z_n)_{E_n} \ (16)$$

Here

$\Phi_n(t)=-\Gamma_n\dot{C}_n(t)+\delta_n(t,C_n)$

By assumptions (6) , (7), $a_0 \neq 0$.

$$\left(R_n(t,\tilde{C}_n) - R_n(t,C_n), z_n\right)_{E_n} + \left(f_n(t,\tilde{C}_n) - f_n(t,C_n), z_n\right)_{E_n} \geq 0$$

Notice, that,

$$\left(g_n(t,\tilde{C}_n) - g_n(t,C_n), z_n\right)_{E_n} = \left(g(x,t,\widetilde{U}) - g(x,t,U), \widetilde{U} - U\right)_S$$

then, estimating the terms on the right hand side of equality (16) using the inequality Cauchy, we obtain

$$\frac{d}{dt}\left((Q_n + \Gamma_n)z_n, z_n\right)_{E_n} \leq 2g_0\left\|\widetilde{U} - U\right\|^2_{L_{2(S)}} + \|z_n\|^2_{E_n} + \|\Phi_n\|^2_{E_n}$$

Integrating the last inequality and taking into account the inequality

$$\|z_n\|^2_{E_n} \leq \frac{1}{q}\left\|\widetilde{U} - U\right\|^2_{\hat{L}_2}$$

we get

$$\frac{d}{dt}\left((Q_n + \Gamma_n)z_n, z_n\right)_{E_n} \leq 2\left(g_0 + \frac{1}{q}\right)\int_0^t \left\|\widetilde{U} - U\right\|^2_{\tau_2} dt + \int_0^t \|\Phi_n\|^2_{E_n} dt +$$

$$+\left((Q_n + \Gamma_n)z_n(0), z_n(0)\right)_{E_n}$$

Moreover, by assumption (14)

$$\left((Q_n + \Gamma_n)z_n, z_n\right)_{E_n} \geq (Q_n z_n, z_n)_{E_n} - \alpha q\|z_n\|^2_{E_n} \geq (1-\alpha)(Q_n z_n, z_n)_{E_n} = (1-\alpha)\left\|\widetilde{U} - U\right\|^2_{\tau_2}, \quad (17)$$

$$\left|\left((Q_n + \Gamma_n)z_n(0), z_n(0)\right)_{E_n}\right| \leq \left(Q_n z_n(0), z_n(0)\right)_{E_n} + \alpha q\|z_n\|^2_{E_n} \leq\leq c(1+\alpha)\left\|\widetilde{U}(x,0) - U(x,0)\right\|^2_{H^1(\Omega)}$$

And by estimate(15)

$$\int_0^t \|\Phi_n\|^2_{E_n} dt \leq 2k\|\Gamma_n\|^2 + 2K\|\delta_n(t,C_n)\|^2_{L_2(0,t,E_n)} \quad (18)$$

Let's denote

$$\int_0^t \left\|\widetilde{U}(x,t) - U(x,t)\right\|^2_{L_2} dt = y_n(t),$$

$$c(1+\alpha)\left\|\widetilde{U}(x,0) - U(x,0)\right\|^2_{H^1(\Omega)} + \quad (19)$$

$$+2k\|\Gamma_n\|^2 + 2\max_{\|C_n\|\leq K}\|\delta_n(t,C_n)\|^2_{L_2(0,t,E_n)} = F_n(t)$$

Then, substituting (17), (18), to(16), we obtain the inequality for$y_n(t)$

$$\frac{dy_n(t)}{dt} \leq C_1 y_n(t) + F_n(t)\ ,\ C_1 = \frac{2\left(g_0 + \frac{1}{q}\right)}{1-\alpha} \quad (20)$$

From which, in turn, by well-known lemma on differential inequalities, the inequality follows [21-24]

$$\frac{dy_n(t)}{dt} \leq e^{C_1 t}$$

From here,

$$\left\|\widetilde{U}(x,t) - U(x,t)\right\|^2_{\tau_2} \leq p_0\|\Delta_n\|^2 + \bar{p}_1\left\|\Gamma^0_n\right\|^2_{E_n} + \quad (21)$$

$$+p_2\|\Gamma_n\|^2 + \bar{p}_3\max_{\|C_n\|\leq K}\delta_n(t,C_n)^2_{L_2(0,T,E_n)}$$

Where constants$\bar{p}_i$ $(i = \overline{0,3})$independent ofn

$$\left\|\tilde{C}_n(t) - C_n(t)\right\|_{E_n}^2 \leq \frac{1}{q}\left\|\widetilde{U} - U\right\|_{L_2}^2 \leq \frac{1}{q}\omega^2$$

where$\omega^2$is the right hand side of inequality (21).

Conclusion

The stability of the approximate solution of the Galerkin method for problem (1) is established under the conditions that (3) - (7) are satisfied and the coordinate system is strongly minimized in space$\hat{L}_2(\Omega)$.

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