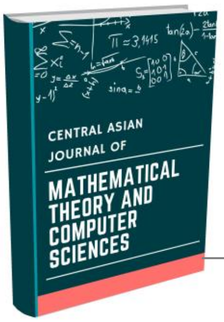

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The Number of Roots of the Egution $a^x = \log_a x$

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Abstract:

this article shows the number of roots of the equation a^x=log_a x, where a>1 and 0<a<1. The derivative of the function is applied to solve this equation. Such information is not found in textbooks. We recommend that the information presented in the article can be used to show students the application of the derivatives in solving equations.

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The domain of the equation $a^x = \log_a x$ (1)

is $x \in (0; \infty)$, where a is a constant such that $a > 0, a \neq 1$. To determine the number of roots of the equation (1), we consider two cases: $a > 1$ and $0 < a < 1$, by applying derivative to both cases. **1- case**. Let $a > 1$. If $a > 1$, then $f(x) = a^x$ and $g(x) = \log_a x$ are incremental functions. In the equation (1), f (x) and g (x) are inverse functions. Let us write the equation (1) in the form $a^{a^x} = x$, (2). If the number x_0 is the root of the equation $a^x = x$, (3), then the equation holds $a^{x_0} = x_0$. In this case, the equation $a^{a^{x_0}} = a^{x_0} = x_0$ is valid and the number, x_0 is considered as the root of the equation (2). We use the following theorem to solve the equation (2).

Theorem [3]. If f (x) is increasing, then $f(x) = x$ and $f(f(x)) = x$ are equally strong equations. We will not dwell on the proof of this theorem. If we take $f(x) = a^x$, then it equals to $f(f(x)) = a^{f(x)} = a^{a^x}$. So according to the above theorem: (2) - and (3) - equations are equally strong. We determine the number of roots of equation (3) when $a > 1$. If we consider logarithm in both parts of equation (3), we get the equation $\ln a = \frac{\ln x}{x}$ (4).

Let us draw a graph of the function $K(x) = \frac{\ln x}{x}$.

1. Area of domain: D (K) = (0; ∞).
2. Derivative. $K'(x) = \frac{1-\ln x}{x^2}$
3. Critical point. $\frac{1-\ln x}{x^2} = 0,\ x = e$.
4. If $x \in (0; e)$, then $K'(x) > 0$, where $K(x)$ is a increasing function. If $x \in (e\,;\,\infty)$, then $K'(x) < 0$. Consequently, $K(x)$ is a decreasing function.

5. At the point $x = e$, $K(x)$ reaches a maximum. $maxK(x) = K(e) = \frac{1}{e}$ is the graph of the function which, is shown in Figure 1. The graph of the function $K(x)$ shows:

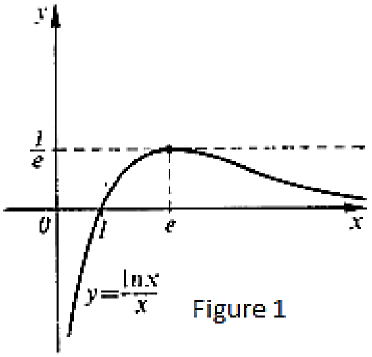


Figure 1

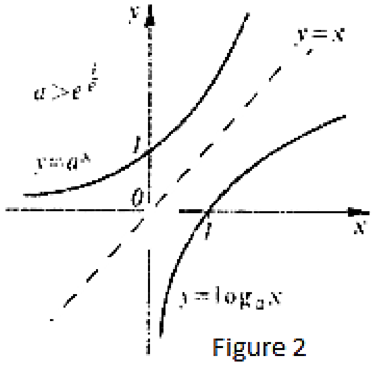


Figure 2

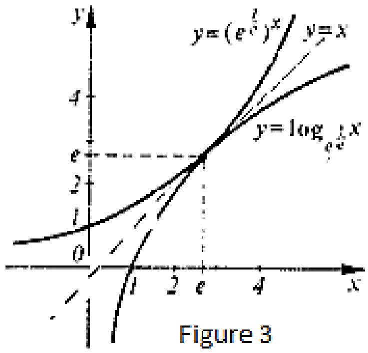


Figure 3

1) If $\ln a > \frac{1}{e}$, that is $a > e^{\frac{1}{e}}$ then the equation (4) has no root.

2) If $\ln a = \frac{1}{e}$, that is $a = e^{\frac{1}{e}}$ then the equation (4) has one root.

3) If $0 < \ln a < \frac{1}{e}$ that is, $1 < a < e^{\frac{1}{e}}$ then the equation (4) has two roots.

Therefore, the equations (3) - and (4) are equivalent equations:

1) If $> e^{\frac{1}{e}}$, the equation (1) has no root (Figure 2).

2) If $= e^{\frac{1}{e}}$, the equation (1) has one root: $x = e$ (Figure 3).

3) If $1 < a < e^{\frac{1}{e}}$ the equation (1) has two roots (Figure 4).

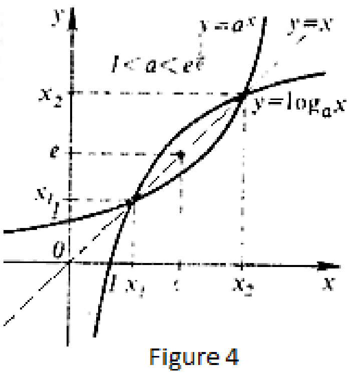


Figure 4

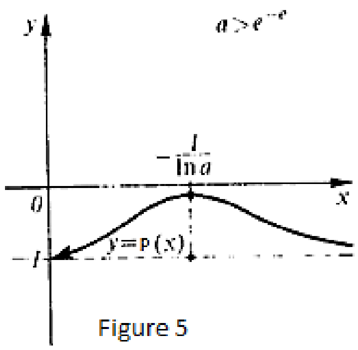


Figure 5

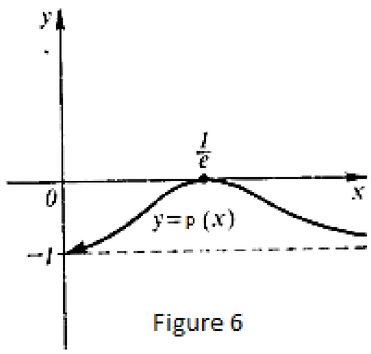


Figure 6

For example: 1) If $a = 1{,}2$, then $(1{,}2)^x = log_{1,2}x$ so, the equation has two roots, due to the $1 < 1{,}2 < e^{\frac{1}{e}}$. The table gives some values of $y = 1{,}2^x$ and $y = log_{1,2}x$ that accept the function. If you look at the table, you can see that this equation has two roots, which these roots lie in the intervals.

x / *funk*	1	1,1	1,25	1,3	1,4	2	10	15
$y = 1{,}2^x$	1,2	1,222	1,255	1,267	1,279	1,44	6,19	15,407
$y = log_{1,2}x$	0	0,522	1,224	1,439	1,640	3,80	12,63	14,85

2) If $a = 2$, then $2^x = log_2 x$ has no root because $2 > e^{\frac{1}{e}}$.

2-case. Let $0 < a < 1$. We answer the question: how many roots does have the equation (1) by checking the function $F(x) = a^x - \log_a x = a^x - \frac{\ln x}{\ln a}$.

1. Area of domain. D (F) = (0; ∞).

2. Derivative: $F'(x) = a^x \ln a - \frac{1}{x \ln a} = \frac{x a^x \ ln^2 a - 1}{x \ln a}$. The function $F'(x)$ takes negative values when the expression $x \ln a$ is in the denominator of function where, x> 0 and 0 <a <1, that is, $x \ln a < 0$, because $0 < a < 1$ results $\ln a < 0$. Therefore, we check the function $p(x) = x a^x ln^2 a - 1$ in the form of a fraction.

Derivative. p′ (x)= $a^x ln^2 a(1 + x \ln a)$. Since, $a^x ln^2 a > 0$ is the sign of p '(x) depends on the sign of $1 + x \ln a$. $1 + x \ln a > 0 \Leftrightarrow x < -\frac{1}{\ln a}$, $1 + x \ln a < 0 \Leftrightarrow x > -\frac{1}{\ln a}$. Hence, the function p (x) is a function that increases in the interval $\left(0\,;\, -\frac{1}{\ln a}\right)$ and decreases in the interval $\left(-\frac{1}{\ln a}\,;\infty\right)$. The function $p(x)$ in $x = -\frac{1}{\ln a}$ reaches its maximum. $p_{max}\left(-\frac{1}{\ln a}\right) = -\frac{\ln a}{e} - 1$.

Hence, when $(0 < a < 1)$ the number of roots of the equation (1) depends on the sign of $-\frac{\ln a}{e} - 1$, because $-\frac{\ln a}{e} - 1 < 0, -\frac{\ln a}{e} - 1 \le 0$ or can also be $-\frac{\ln a}{e} - 1 > 0$.

If $p_{max} \le 0$ is for all $x > 0$, then the function $F(x)$ increases in the range $x > 0$ and has a single root x_0. The graph of the function p (x) is shown in Figures 5 and 6, and the graph of the function F (x) is shown in Figures 7 and 8.

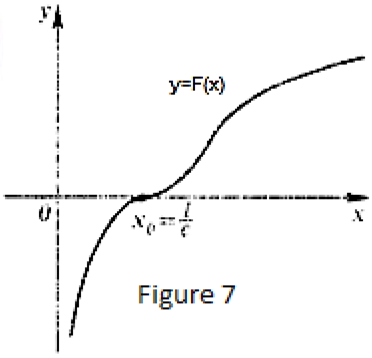


Figure 7

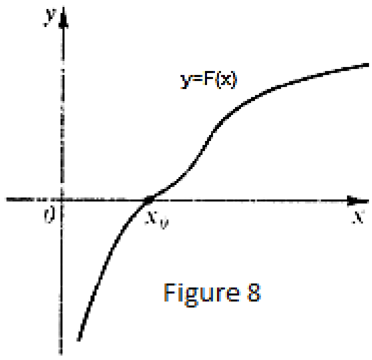


Figure 8

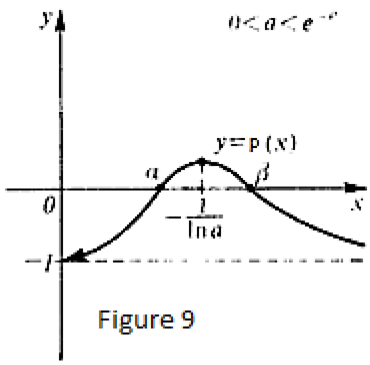


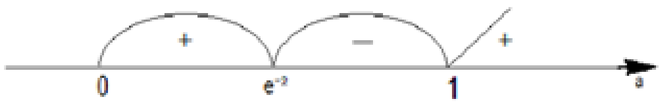
Figure 9

Solve the inequality $p_{max}\left(-\frac{1}{\ln a}\right) = -\frac{\ln a}{e} - 1 \le 0$. $\ln a \ge -e$, $a \ge e^{-e}$. Therefore, if $e^{-e} \le a < 1$, then the equation (1) has a single root x_0 . (see 5-6 figure). If $0 < a < e^{-e}$,then $p_{max} > 0$, and the graph of the function p (x) is as shown in Figure 9. To substantiate this, it is sufficient to know that $p(0) = -1$ and $p(1) < 0$.

If $0 < a < e^{-e}$, we show that $p(1) = a ln^2 a - 1 < 0$. To do this, we consider the function $g(a) = a ln^2 a$. Derivative. $g'(a) = \ln a(\ln a + 2)$.

Critical points:$\ln a(\ln a + 2) = 0$ 1) $\ln a = 0$, $a = 1$.2) $\ln a + 2 = 0$, $a = e^{-2}$. Hence, if $a \in (0\,;\, e^{-2}) \cup (1\,;\infty)$, $g'(a) > 0$ in these intervals the function $g(a)$ is a increasing function. If $a \in (e^{-2}\,; 1)$ then $g'(a) < 0$ in which the function $g(a)$ is a decreasing function. $a = e^{-2}$ is the maximum point of the function, $a = 1$ is the minimum point of the function. (see figure 10). Hence, the function $p(x)$ becomes zero at two points α and β, where $0 < \alpha < -\frac{1}{\ln a}$ and $-\frac{1}{\ln a} < \beta < 1$. Hence, the function $F(x)$ is increasing in the interval $(0\,;\, \alpha)$, decreasing in the interval $(\alpha\,; \beta)$, and increasing in the interval $(\beta; 1)$.

Therefore, at $x = \alpha$ the function $F(x)$ reaches a maximum, and at $x = \beta$ the function $F(x)$ reaches a minimum. The graph of the function $F(x)$ is shown in Figure 11. Now we show that



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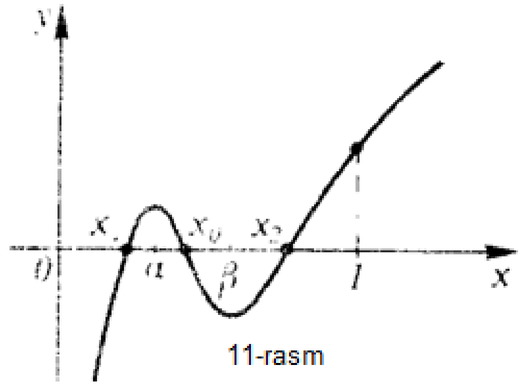

11-rasm

$p(x_0) = x_0 a^{x_0} ln^2 a - 1 = x_0^2\, ln^2 a - 1 = ln^2 a^{x_0} - 1 = ln^2 x_0 - 1.$

If $a \in (0\,;\, e^{-e})$, then $x_0 < \frac{1}{e}$. Suppose the opposite, let $x_0 > \frac{1}{e}$.

Then $x_0 = a^{x_0} < a^{\frac{1}{e}} < \frac{1}{e}$. This contradiction indicates that $x_0 < \frac{1}{e}$. Since $\ln x_0 < -1$, we find that $p\,(x_0) > 0$, because of the inequality $F'(x_0) < 0,\ \alpha < x_0 < \beta$ is valid. It follows that $F(\alpha) > 0$ and $F(\beta) < 0$. From the monotony of the function $F(x)$ it follows that this function has one root in the intervals $(0;\alpha)$, $(\alpha;\,\beta)$ and $(\beta;1)$ and has no root in the interval $(1;\infty)$. Therefore, the equation (1) has three roots, if $0 < a < \frac{1}{e^e}$. And if $\frac{1}{e^e} \le a < 1$, it has one root. ($\frac{1}{e^e} \approx 0{,}0659$).

For example: let $a = \frac{1}{16}$. In this case the equation$\left(\frac{1}{16}\right)^x = log_{\frac{1}{16}} x$ has three roots. Because, $0 < \frac{1}{16} < \frac{1}{e^e}$.

1) Let $= \frac{1}{2}$. $\left(\frac{1}{16}\right)^{\frac{1}{2}} = log_{\frac{1}{16}} \frac{1}{2} = \frac{1}{4}$. $x = \frac{1}{4}$, so, $\left(\frac{1}{16}\right)^{\frac{1}{4}} = log_{\frac{1}{16}} \frac{1}{4} = \frac{1}{2}$, The exact value of the third root cannot be specified. Let this root is approximately equal to the root of the equation $\left(\frac{1}{16}\right)^x = x$.

2) Let $a = \frac{1}{64}$. Then $\left(\frac{1}{64}\right)^x = log_{\frac{1}{64}} x$ has three roots. Because $0 < \frac{1}{64} < \frac{1}{e^e}$.

3) Let $a = \frac{1}{10}$. Then $\left(\frac{1}{10}\right)^x = log_{\frac{1}{10}} x$ has a single root. Because $\frac{1}{e^e} < \frac{1}{10} < 1$.

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